Problem 3. Simplest model of gas discharge

Solution

Part A. Non-self-sustained gas discharge

A1. Let us derive an equation describing the change of the electron number density with time. It is determined by the two processes; the generation of ion pairs by external ionizer and the recombination of electrons with ions. At ionization process electrons and ions are generated in pairs, and at recombination processthey disappear in pairs as well.Thus, their concentrations are alwaysequal at any given time, i.e.

$$n(t) = n_e(t) = n_i(t) \quad \text{(A1.1).}$$

Then the equation describing the numberdensityevolution of electrons and ions in time can be written as

$$\frac{dn(t)}{dt} = Z_{ext} - rn(t)^2 \quad \text{(A1.2).}$$

It is easy to show that at$t \to 0$ the function $\tanh bt \to 0$, therefore, by virtue of the initial condition $n(0) = 0$,one finds

$$n_0 = 0 \quad \text{(A1.3).}$$

Substituting $n_e(t) = a \tanh bt$ in (A1.2) and separating it in the independent functions (hyperbolic, or 1 ande^x), one gets

$$a = \sqrt{\frac{Z_{ext}}{r}} \quad \text{(A1.4),}$$

$$b = \sqrt{rZ_{ext}} \quad \text{(A1.5).}$$

A2. According to equation (A1.4) the number density of electronsat steady-state is expressed in terms of the external ionizer activity as

$$n_{e1} = \sqrt{\frac{Z_{ext\,1}}{r}} \quad \text{(A2.1),}$$

$$n_{e2} = \sqrt{\frac{Z_{ext\,2}}{r}} \quad \text{(A2.2),}$$

$$n_e = \sqrt{\frac{Z_{ext\,1} + Z_{ext\,2}}{r}} \quad \text{(A2.3).}$$

Thus,the following analogue of the Pythagorean theorem is obtained as

$$n_e = \sqrt{n_{e1}^2 + n_{e2}^2} = 20.0 \cdot 10^{10}\,\text{cm}^{-3}. \quad \text{(A2.4)}$$

A3. In the steady state, the balance equations of electrons and ions in the tube volume take the form

$$Z_{ext}SL = rn_e n_i SL + \frac{I_e}{e} \quad \text{(A3.1),}$$

$$Z_{ext}SL = rn_e n_i SL + \frac{I_i}{e} \quad \text{(A3.2).}$$

It follows from equations (A3.1) and (A3.2) that the ion and electron currents are equal, i.e.

$$I_e = I_i \quad \text{(A3.3).}$$

At the same time the total current in each tube section is the sum of the electron and ion currents

$$I = I_e + I_i \quad \text{(A3.4).}$$

By definition ofthe current density the following relations hold

$$I_e = \frac{I}{2} = en_e vS = e\beta n_e ES \quad \text{(A3.5),}$$

$$I_i = \frac{I}{2} = en_i vS = e\beta n_i ES \quad \text{(A3.6).}$$

Substituting (A3.5) and (A3.6) into (A3.1) and (A3.2), the following quadratic equation for the current is derived

$$Z_{ext}SL = rSL\left(\frac{I}{2e\beta ES}\right)^2 + \frac{I}{2e} \quad \text{(A3.7).}$$

The electric field strength in the gas is equal to

$$E = \frac{U}{L} \quad \text{(A3.8).}$$

and solution to the quadratic equation (A3.7) takes the form

$$I = \frac{e\beta^2 U^2 S}{rL^3}\left(-1 \pm \sqrt{1 + \frac{4rZ_{ext}L^4}{\beta^2 U^2}}\right) \quad \text{(A3.9).}$$

It is obvious that only positive root does make sense, i.e.

$$I = \frac{e\beta^2 U^2 S}{rL^3}\left(\sqrt{1+\frac{4rZ_{ext}L^4}{\beta^2 U^2}}-1\right) \quad \text{(A3.10).}$$

A4. At low voltages (A3.10) simplifies and gives the following expression

$$I = 2Ue\beta\sqrt{\frac{Z_{ext}}{r}}\frac{S}{L}. \quad \text{(A4.1)}$$

which is actually the Ohm law.

Using the well-known relation

$$R = \frac{U}{I} \quad \text{(A4.2)}$$

together with

$$R = \rho\frac{L}{S} \quad \text{(A4.3),}$$

one gets

$$\rho = \frac{1}{2e\beta}\sqrt{\frac{r}{Z_{ext}}} \quad \text{(A4.4).}$$

Part B. Self-sustained gas discharge

B1. Consider a gas layer located between x and $x+dx$. The rate of change in the electron number inside the layer due to the electric current is given for a small time interval dt by

$$dN_e^I = \frac{I_e(x+dx)-I_e(x)}{e}dt = \frac{1}{e}\frac{dI_e(x)}{dx}dxdt. \quad \text{(B1.1).}$$

This change is due to the effect of the external ionization and the electron avalanche formation.

The external ionizer creates the following number of electrons in the volume Sdx

$$dN_e^{ext} = Z_{ext}Sdxdt \quad \text{(B1.2).}$$

whereas the electron avalanche produces the number of electrons found as

$$dN_e^a = \alpha N_e dl = n_e Sdxvdt = \alpha\frac{I_e(x)}{e}dxdt \quad \text{(B1.3).}$$

The balance equation for the number of electrons is written as

$$dN_e^I = dN_e^{ext} + dN_e^a \quad \text{(B1.4),}$$

which results in the following differential equation for the electron current

$$\frac{dI_e(x)}{dx} = eZ_{ext}S + \alpha I_e(x) \quad \text{(B1.5).}$$

On substituting $I_e(x) = C_1 e^{A_1 x} + A_2$, one derives

$$A_1 = \alpha \quad \text{(B1.6),}$$

$$A_2 = -\frac{eZ_{ext}S}{\alpha} \quad \text{(B1.7).}$$

B2. Given the fact that the ions flow in the direction opposite to the electron motion, the balance equation for the number of ions is written as

$$dN_i^I = dN_i^{ext} + dN_i^a \quad \text{(B2.1),}$$

where

$$dN_i^I = \frac{I_i(x)-I_i(x+dx)}{e}dt = -\frac{1}{e}\frac{dI_i(x)}{dx}dxdt \quad \text{(B2.2).}$$

$$dN_i^{ext} = Z_{ext}Sdxdt \quad \text{(B2.3).}$$

$$dN_i^a = \alpha\frac{I_e(x)}{e}dxdt \quad \text{(B2.4).}$$

Hence, the following differential equation for the ion current is obtained

$$-\frac{dI_i(x)}{dx} = eZ_{ext}S + \alpha I_e(x). \quad \text{(B2.5)}$$

On substituting the previously found electron current together with the ion current, $I_i(x) = C_2 + B_1 e^{B_2 x}$, yields

$$B_1 = -C_1 \quad \text{(B2.6),}$$

$$B_2 = \alpha \quad \text{(B2.7).}$$

B3. Since the ions starts to move from the anode located at $x = L$, the following condition holds

$$I_i(L) = 0 \quad \text{(B3.1).}$$

B4. By definition of secondary electron emission coefficient the following condition should be imposed

$$I_e(0) = \gamma I_i(0) \quad \text{(B4.1).}$$

B5. Total current in each tube section is the sum of the electron and ion currents:

$$I = I_e + I_i = C_2 - \frac{eZ_{ext}S}{\alpha} \qquad \text{(B5.1).}$$

Aftersubstituting the boundary conditions (B3.1) and (B4.1):

$$C_2 - C_1 e^{\alpha L} = 0 \qquad \text{(B5.2)}$$

and

$$C_1 - \frac{eZ_{ext}S}{\alpha} = \gamma(C_2 - C_1) \qquad \text{(B5.3).}$$

Solving (B5.2) and (B5.3) one can obtain:

$$C_2 = \frac{eZ_{ext}S}{\alpha}\left(\frac{1}{e^{-\alpha L}(1+\gamma)-\gamma}\right) \qquad \text{(B5.4).}$$

So the total current:

$$I = \frac{eZ_{ext}S}{\alpha}\left(\frac{1}{e^{-\alpha L}(1+\gamma)-\gamma} - 1\right) \qquad \text{(B5.5).}$$

B6. When the discharge gap length is increased, the denominator in formula (B5.1) decreases. At that moment, when it turns zero, the electric current in the gas becomes self-sustaining and external ionizer can be turned off. Thus,

$$L_{cr} = \frac{1}{\alpha}\ln\left(1 + \frac{1}{\gamma}\right) \qquad \text{(B6.1).}$$