Problem 3. Simplest model of gas discharge

☐ This problem is a tribute to my teacher in science, Prof. F. Baimbetov
Part A. Non self-sustained gas discharge

An external ionizer creates $Z_{\text{ext}}$ pairs of singly ionized ions and free electrons per unit volume and per unit time.

The number of recombining events $Z_{\text{rec}}$ that occurs in the gas per unit volume and per unit time is given by

$$Z_{\text{rec}} = r n_e n_i$$
At time $t = 0$ the external ionizer is switched on and the initial number densities of electrons and ions in the gas are both equal to zero. The electron number density $n_e(t)$ depends on time $t$ as follows:

$$n_e(t) = n_0 + a \tanh bt$$

Find $n_0, a, b$ and express them in terms of $Z_{ext}$ and $r$.

$$n(t) = n_e(t) = n_i(t)$$

$$\frac{dn(t)}{dt} = Z_{ext} - rn(t)^2$$

$$n_e(t) = n_0 + a \tanh bt$$

$t \to 0$
$tanh \, bt \to 0$

$n_0 = 0$

$$n_e(t) = a \tanh bt$$

$$a = \sqrt{\frac{Z_{ext}}{r}}$$

$$b = \sqrt{rZ_{ext}}$$
Question A2

A2 Find the electron number density $n_e$ at equilibrium when both external ionizers are switched on simultaneously.

\[
\frac{dn(t)}{dt} = Z_{\text{ext}} - r n(t)^2
\]

\[
n_e = \sqrt[2]{\frac{Z_{\text{ext1}}}{r}}
\]

\[
n_{e2} = \sqrt[2]{\frac{Z_{\text{ext2}}}{r}}
\]

\[
n_e = \sqrt[2]{\frac{Z_{\text{ext1}} + Z_{\text{ext2}}}{r}}
\]

\[
n_e = \sqrt{n_{e1}^2 + n_{e2}^2} = 20.0 \cdot 10^{10} \text{sm}^{-3}
\]
A3 Express the electric current $I$ in the tube in terms of $U, b, L, S, Z_{ext}, r$.

\[ v = bE \]
\[ b_e = b_i \]

\[ Z_{ext}SL = rn_e n_i SL + \frac{I_e}{e} \]
\[ Z_{ext}SL = rn_e n_i SL + \frac{I_i}{e} \]

\[ I_e = \frac{l}{2} = en_e vS = ebn_e ES \]
\[ I_i = \frac{l}{2} = en_i vS = ebn_i ES \]

\[ Z_{ext}SL = rSL \left( \frac{l}{2ebES} \right)^2 + \frac{l}{2e} \]

\[ I = \frac{eb^2U^2S}{rL^3} \left( \sqrt{1 + \frac{4rZ_{ext}L^4}{b^2U^2}} - 1 \right) \]
Question A4

**A4** Find the resistivity $\rho_{gas}$ of the gas at sufficiently small values of the voltage applied.

\[ I = \frac{eb^2U^2S}{rL^3} \left( \sqrt{1 + \frac{4rZ_{ext}L^4}{b^2U^2}} - 1 \right) \]

\[ U \rightarrow 0 \]

\[ I = U2eb \sqrt{\frac{Z_{ext}S}{rL}} \]

\[ R = \frac{U}{I} \]
\[ R = \rho \frac{L}{S} \]

\[ \rho = \frac{1}{2eb \sqrt{Z_{ext}}} \]
Part B. Self-sustained gas discharge

Attention! In the sequel assume that the external ionizer continues to operate, neglect the electric field due to the charge carriers such that the electric field is uniform along the tube, and the recombination can be completely ignored.
$I_e(x) = C_1 e^{A_1 x} + A_2$

Find $A_1, A_2$ and express them in terms of $Z_{ext}, \alpha, e, L, S$. 

\[
\frac{dN_e}{dx} = \frac{I_e(x+dx)-I_e(x)}{e} = \frac{1}{e} \frac{dl_e(x)}{dx} \cdot dx \cdot dt
\]

\[
dN_e^{ext} = z_{ext} S \cdot dx \cdot dt
\]

\[
dN_e^l = dN_e^{ext} + dN_e^a
\]

\[
\frac{dl_e(x)}{dx} = eZ_{ext}S + \alpha I_e(x)
\]

\[
A_1 = \alpha
\]

\[
A_2 = -\frac{eZ_{ext}S}{\alpha}
\]
Question B2

\[ I_i(x) = C_2 + B_1 e^{B_2 x} \]

Find \( B_1, B_2 \) and express them in terms of \( Z_{\text{ext}}, \alpha, e, L, S, \hat{C}_1 \).

\[
\frac{dN_i^l}{dx} = \frac{I_i(x) - I_i(x)}{e} = -e \frac{dI_i(x)}{dx} dt
\]

\[
dN_i^a = \alpha \frac{I_{\text{e}}(x)}{e} dxdt
\]

\[
dN_i^\text{ext} = Z_{\text{ext}} S dxdt
\]

\[
\frac{dN_i^l}{dx} = dN_i^\text{ext} + dN_i^a
\]

\[
-e \frac{dI_i(x)}{dx} = eZ_{\text{ext}} S + \alpha I_{\text{e}}(x)
\]

\[
B_1 = -C_1, \quad B_2 = \alpha
\]
**Question B3-B5**

**B3** Write down the condition for \( I_i(x) \) at \( x = L \).

**B4** Write down the condition for \( I_i(x) \) and \( I_e(x) \) at \( x = 0 \).

**B5** Find the total current \( I \) and express it in terms of \( Z_{ext}, \alpha, \gamma, e, L, S \).

\[
I_e(0) = \gamma I_i(0)
\]

\[
I_i(L) = 0
\]

\[
I_e(x) = C_1 e^{A_1 x} + A_2
\]

\[
I_i(x) = C_2 + B_1 e^{B_2 x}
\]

\[
I = I_e(x) + I_i(x) = \frac{e Z_{ext} S}{\alpha \left[ e^{-\alpha L} - \frac{\gamma}{\gamma + 1} \right]}
\]
Let the Townsend coefficient $\alpha$ be constant. When the length of the tube turns out greater than some critical value, i.e. $L > L_{cr}$, the external ionizer can be turned off and the discharge becomes self-sustained.

B6 Find $L_{cr}$ and express it in terms of $Z_{ext}, \alpha, \gamma, e, L, S$.

$\alpha = \alpha_1 p \exp\left(-\frac{\alpha_2 p}{E}\right)$

B7 Find and evaluate the minimum voltage $U_{\text{min}}$ applied across the plates, at which the self-sustained gas discharge can still appear in neon.

$L_{cr} = \frac{1}{\alpha} \ln \left(1 + \frac{1}{\gamma}\right)$

$U = \frac{\alpha_2 p L}{\ln \left(\frac{\alpha_1 p L}{\ln(1+1/\gamma)}\right)}$

$\frac{dU}{dz} = 0$

$z_{\text{min}} = \frac{e}{\alpha_1} \ln(1 + 1/\gamma)$

$U_{\text{min}} = \frac{e\alpha_1}{\alpha_2} \ln \left(1 + \frac{1}{\gamma}\right) = 122B$