The Greenlandic Ice Sheet

## Solutions

### 3.1
The pressure is given by the hydrostatic pressure $p(x, z) = \rho_{\text{ice}} g (H(x) - z)$, which is zero at the surface.

### 3.2a
The outward force on a vertical slice at a distance $x$ from the middle and of a given width $\Delta y$ is obtained by integrating up the pressure times the area:

$$F(x) = \Delta y \int_0^{H(x)} \rho_{\text{ice}} g \, (H(x) - z) \, dz = \frac{1}{2} \Delta y \rho_{\text{ice}} g H(x)^2$$

which implies that $\Delta F = F(x) - F(x + \Delta x) = -\frac{dF}{dx} \Delta x = -\Delta y \rho_{\text{ice}} g H(x) \frac{dH}{dx} \Delta x$.

This finally shows that

$$S_b = \frac{\Delta F}{\Delta x \Delta y} = -\rho_{\text{ice}} g H(x) \frac{dH}{dx}$$

Notice the sign, which must be like this, since $S_b$ was defined as positive and $H(x)$ is a decreasing function of $x$.

### 3.2b
To find the height profile, we solve the differential equation for $H(x)$:

$$-\frac{S_b}{\rho_{\text{ice}} g} = H(x) \frac{dH}{dx} = \frac{1}{2} \frac{d}{dx} H(x)^2$$

with the boundary condition that $H(L) = 0$. This gives the solution:

$$H(x) = \sqrt{\frac{2S_b L}{\rho_{\text{ice}} g}} \sqrt{1 - \frac{x}{L}}$$

Which gives the maximum height $H_m = \sqrt{\frac{2S_b L}{\rho_{\text{ice}} g}}$.

Alternatively, dimensional analysis could be used in the following manner. First notice that $\mathcal{L} = [H_m] = [\rho_{\text{ice}} g^\alpha \tau_b^\gamma \mathcal{L}^\delta]$. Using that $[\rho_{\text{ice}} g] = \mathcal{M} \mathcal{T}^{-2}$, $[\tau_b] = \mathcal{M} \mathcal{T}^{-2}$, demands that $\mathcal{L} = [H_m] = [\rho_{\text{ice}} g^\alpha \tau_b^\gamma \mathcal{L}^\delta] = \mathcal{M}^{\alpha + \gamma} \mathcal{T}^{-3 \alpha + \gamma + 2 \delta - 2 \gamma}$, which again implies $\alpha + \gamma = 0$, $-3 \alpha + \beta - \gamma + \delta = 1$, $2 \beta + 2 \gamma = 0$. These three equations are solved to give $\alpha = -\gamma = \delta - 1$, which shows that

$$H_m \propto \left( \frac{S_b}{\rho_{\text{ice}} g} \right)^{\gamma} L^{1-\gamma}$$

Since we were informed that $H_m \propto \sqrt{L}$, it follows that $\gamma = 1/2$. With the boundary condition $H(L) = 0$, the solution then take the form

$$H(x) \propto \left( \frac{S_b}{\rho_{\text{ice}} g} \right)^{1/2} \sqrt{L - x}$$

The proportionality constant of $\sqrt{L}$ cannot be determined in this approach.
### 3.2.3 For the rectangular Greenland model, the area is equal to $A = 10L^2$ and the volume is found by integrating up the height profile found in problem 3.2b:

$$V_{G,\text{ice}} = (5L^2) \int_0^L H(x) \, dx = 10L \int_0^L \left( \frac{x_b L}{\rho_{\text{ice}} g} \right)^{1/2} \sqrt{1 - x/L} \, dx = 10H_m L^2 \int_0^1 \sqrt{1 - \frac{x}{L}} \, dx$$

$$= 10H_m L^2 \left[ -\frac{2}{3} (1 - \frac{x}{L})^{3/2} \right]_0^1 = \frac{20}{3} H_m L^2 \propto L^{5/2},$$

where the last line follows from the fact that $H_m \propto \sqrt{L}$. Note that the integral need not be carried out to find the scaling with $L$. This implies that $V_{G,\text{ice}} \propto A_G^{5/4}$ and the wanted exponent is $\gamma = 5/4$.

### 3.3 According to the assumption of constant accumulation $c$ the total mass accumulation rate from an area of width $\Delta y$ between the ice divide at $x = 0$ and some point at $x > 0$ must equal the total mass flux through the corresponding vertical cross section at $x$. That is: $\rho c x \Delta y = \rho \Delta y H_m v_x(x)$, from which the velocity is isolated:

$$v_x(x) = \frac{c x}{H_m}$$

### 3.4 From the given relation of incompressibility it follows that

$$\frac{d v_z}{d z} = -\frac{d v_x}{d x} = - \frac{c}{H_m}$$

Solving this differential equation with the initial condition $v_z(0) = 0$, shows that:

$$v_z(z) = - \frac{c z}{H_m}$$

### 3.5 Solving the two differential equations

$$\frac{d z}{d t} = - \frac{c z}{H_m} \quad \text{and} \quad \frac{d x}{d t} = \frac{c x}{H_m}$$

with the initial conditions that $z(0) = H_m$, and $x(0) = x_i$ gives

$$z(t) = H_m e^{-ct/H_m} \quad \text{and} \quad x(t) = x_i e^{ct/H_m}$$

This shows that $z = H_m x_i / x$, meaning that flow lines are hyperbolas in the $xz$-plane. Rather than solving the differential equations, one can also use them to show that

$$\frac{d}{d t} (xz) = \frac{d x}{d t} z + x \frac{d z}{d t} = \frac{c x}{H_m} z - x \frac{c z}{H_m} = 0$$

which again implies that $xz = \text{const}$. Fixing the constant by the initial conditions, again leads to the result that $z = H_m x_i / x$.

### 3.6 At the ice divide, $x = 0$, the flow will be completely vertical, and the $t$-dependence of $z$ found in 3.5 can be inverted to find $\tau(z)$. One finds that $\tau(z) = \frac{H_m}{c} \ln \left( \frac{H_m}{z} \right)$.
The present interglacial period extends to a depth of 1492 m, corresponding to 11,700 year. Using the formula for $\tau(z)$ from problem 3.6, one finds the following accumulation rate for the interglacial:

$$c_{ig} = \frac{H_m}{11,700 \text{ years}} \ln \left(\frac{H_m}{H_m - 1492 \text{ m}}\right) = 0.1749 \text{ m/year}.$$ 

The beginning of the ice age 120,000 years ago is identified as the drop in $\delta^{18}O$ in figure 3.2b at a depth of 3040 m. Using the vertical flow velocity found in problem 3.4, one has $\frac{dz}{z} = -\frac{c}{H_m} dt$, which can be integrated down to a depth of 3040 m, using a stepwise constant accumulation rate:

$$H_m \ln \left(\frac{H_m}{H_m - 3040 \text{ m}}\right) = -H_m \int_{H_m}^{H_m = 3040 \text{ m}} \frac{1}{z} dz$$

$$= \int_{11,700 \text{ year}}^{120,000 \text{ year}} c_{ia} \ dt + \int_{0}^{11,700 \text{ year}} c_{ig} \ dt$$

$$= c_{ia}(120,000 \text{ year} - 11,700 \text{ year}) + c_{ig} 11,700 \text{ year}$$

Isolating from this equation leads to $c_{ia} = 0.1232$, i.e. far less precipitation than now.

Reading off from figure 3.2b: $\delta^{18}O$ changes from $-43.5 \%$ to $-34.5 \%$. Reading off from figure 3.2a, $T$ then changes from $-40 ^\circ C$ to $-28 ^\circ C$. This gives $\Delta T \approx 12 ^\circ C$.

From the area $A_G$ one finds that $L = \sqrt{A_G/10} = 4.14 \times 10^5 \text{ m}$. Inserting numbers in the volume formula found in 3.2c, one finds that:

$$V_{G,\text{ice}} = \frac{20}{3} L^{5/2} \frac{2S_b}{\rho_{\text{ice}} g} = 3.45 \times 10^{15} \text{ m}^3$$

This ice volume must be converted to liquid water volume, by equating the total masses, i.e. $V_{G,\text{wa}} = V_{G,\text{ice}} \frac{\rho_{\text{ice}}}{\rho_{\text{wa}}} = 3.17 \times 10^{15} \text{ m}^3$, which is finally converted to a sea level rise, as $h_{G,\text{rise}} = \frac{V_{G,\text{wa}}}{A_o} = 8.79 \text{ m}$. 
The total mass of the ice is

\[ M_{\text{ice}} = V_{\text{G,ice}} \rho_{\text{ice}} = 3.17 \times 10^{18} \text{ kg} = 5.31 \times 10^{-7} m_E \]

The total gravitational potential felt by a test mass \( m \) at a certain height \( h \) above the surface of the Earth, and at a polar angle \( \theta \) (cf. figure 3.S1), with respect to a rotated polar axis going straight through the ice sphere is found by adding that from the Earth with that from the ice:

\[
U_{\text{tot}} = -\frac{G m_E m}{R_E + h} - \frac{G M_{\text{ice}} m}{r} = -m g R_E \left( \frac{1}{1 + h/R_E} + \frac{M_{\text{ice}}/m_E}{r/R_E} \right)
\]

where \( g = G m_E / R_E^2 \). Since \( h/R_E \ll 1 \) one may use the approximation given in the problem, \((1 + x)^{-1} \approx 1 - x, \ |x| \ll 1\), to approximate this by

\[
U_{\text{tot}} \approx -m g R_E \left( 1 - \frac{h}{R_E} + \frac{M_{\text{ice}}/m_E}{r/R_E} \right).
\]

Isolating \( h \) now shows that \( h = h_0 + \frac{M_{\text{ice}}/m_E}{r/R_E} R_E \), where \( h_0 = R_E + U_{\text{tot}}/(mg) \). Using again that \( h/R_E \ll 1 \), trigonometry shows that \( r \approx 2 R_E |\sin(\theta/2)| \), and one has:

\[
h(\theta) - h_0 \approx \frac{M_{\text{ice}}/m_E}{2|\sin(\theta/2)|} \frac{1.69 \text{ m}}{|\sin(\theta/2)|}.
\]

To find the magnitude of the effect in Copenhagen, the distance of 3500 km along the surface is used to find the angle \( \theta_{\text{CPH}} = (3.5 \times 10^6 \text{ m})/R_E \approx 0.549 \), corresponding to \( h_{\text{CPH}} - h_0 \approx 6.25 \text{ m} \). Directly opposite to Greenland corresponds to \( \theta = \pi \), which gives \( h_{\text{OPP}} - h_0 \approx 1.69 \text{ m} \). The difference is then \( h_{\text{CPH}} - h_{\text{OPP}} \approx 4.56 \text{ m} \), where \( h_0 \) has dropped out.
Approach with forces:

This problem can also be solved using forces. The basic equations for mechanical equilibrium of the test particle is then a simple matter of balancing the two gravitational forces, $\vec{F}_E$ and $\vec{F}_G$, with the reaction force from the Earth, $\vec{F}_R$. Given the angles indicated in Figure 3.S2, the force balance along locally vertical and horizontal directions, respectively, read

$$F_E + F_G \cos(\delta) = F_R \cos(\phi)$$

and

$$F_G \sin(\delta) = F_R \sin(\phi)$$

which can be divided to obtain (using that $\delta = \pi/2 - \theta/2$):

$$\tan(\phi) = \frac{F_G \sin(\delta)}{F_E + F_G \cos(\delta)} = \frac{F_G \cos(\theta/2)}{1 + (F_G/F_E) \sin(\theta/2)}$$

where we have plugged in the gravitational forces and the relevant distances. We have also
approximated the fraction, using that $M_{\text{ice}}/m_E = 5.31 \times 10^{-7} \ll 1$, which is only valid not too close to Greenland, i.e. for a certain size of $\theta$. Since the local sea surface will be perpendicular to the reaction force, it is seen from figure 3.S2 that

$$\tan(\varphi) = \frac{dh}{dx} = \frac{dh}{d\theta} = \frac{1}{R_E} \frac{dh}{d\theta}$$

whereby

$$\frac{dh}{d\theta} = R_E \frac{M_{\text{ice}}/m_E}{4 \sin^2(\theta/2)} \cos(\theta/2)$$

The difference in sea levels in Copenhagen and opposite to Greenland can now be obtained by integrating this expression. That is

$$h_{\text{CPH}} - h_{\text{OPP}} = R_E \frac{M_{\text{ice}}}{m_E} \int_{\pi}^{\theta_{\text{CPH}}} \frac{\cos(\theta/2)}{4 \sin^2(\theta/2)} \, d\theta$$

$$= R_E \frac{M_{\text{ice}}}{2 m_E} \int_{\sin(\theta_{\text{CPH}}/2)}^{1} q^{-2} \, dq$$

$$= R_E \frac{M_{\text{ice}}}{2 m_E} \left( \frac{1}{\sin(\theta_{\text{CPH}}/2)} - 1 \right)$$

where we have made the substitution $q = \sin(\theta/2)$. Plugging in the numbers found above, we obtain again $h_{\text{CPH}} - h_{\text{OPP}} \approx 4.56$. Note that this solution strategy necessarily involves consideration of tangential force components alongside with the radial components.