Solutions

A single spherical silver nanoparticle

<table>
<thead>
<tr>
<th></th>
<th>Volume of the nanoparticle: $V = \frac{4}{3} \pi R^3 = 4.19 \times 10^{-24} \text{ m}^3$.</th>
<th>0.7</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mass of the nanoparticle: $M = V \rho_{Ag} = 4.39 \times 10^{-20} \text{ kg}$ .</td>
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<td>Number of ions in the nanoparticle: $N = \frac{M}{M_{Ag}} = 2.45 \times 10^5$ .</td>
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<tr>
<td>2.1</td>
<td>Charge density $\rho = \frac{eN}{V} = 9.38 \times 10^9 \text{ C m}^{-3}$ , charge density $\rho = en$.</td>
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<tr>
<td></td>
<td>Electrons’ concentration $n = \frac{N}{V} = 5.85 \times 10^{28} \text{ m}^{-3}$.</td>
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<tr>
<td></td>
<td>Total charge of free electrons $Q = eN = 3.93 \times 10^{-14} \text{ C}$.</td>
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<tr>
<td></td>
<td>Total mass of free electrons $m_0 = m_eN = 2.23 \times 10^{-25} \text{ kg}$.</td>
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</tr>
</tbody>
</table>

The electric field in a charge-neutral region inside a charged sphere

|               | For a sphere with radius $R$ and constant charge density $\rho$, for any point inside the sphere designated by radius-vector $\mathbf{r} = r\mathbf{e}_r$ ($r < R$) Gauss's law yields directly $4\pi r^2 \varepsilon_0 E_+ = \frac{4}{3} \pi r^3 \rho \mathbf{e}_r$, where $\mathbf{e}_r$ is the unit radial vector pointing away from the center of the sphere. Thus, $E_+ = \frac{\rho}{3\varepsilon_0} \mathbf{r}$. | 1.2 |
| 2.2           | Likewise, inside another sphere of radius $R_1$ and charge density $-\rho$ the field is $E_- = \frac{-\rho}{3\varepsilon_0} \mathbf{r}'$, where $\mathbf{r}'$ is the radius-vector of the point in the coordinate system with the origin in the center of this sphere. Superposition of the two charge configurations gives the setup we want with $\mathbf{r}' = \mathbf{r} - x_d$. So inside the charge-free region $|\mathbf{r} - x_p| < R_1$ the field is $E = E_+ + E_- = \frac{\rho}{3\varepsilon_0} \mathbf{r} + \frac{-\rho}{3\varepsilon_0} (\mathbf{r} - x_d)$ or $E = \frac{\rho}{3\varepsilon_0} x_d$ with pre-factor $A = \frac{1}{3}$. |     |

The restoring force on the displaced electron cloud

|               | With $x_p = x_p \mathbf{e}_x$ and $x_p \ll R$ we have from above that approximately the field induced inside the particle is $E_{\text{ind}} = \frac{\rho}{3\varepsilon_0} x_p$. The number of electrons on the particle’s border that produced $E_{\text{ind}}$ is negligibly smaller than the number of electrons inside the particle, so $F \approx QE_{\text{ind}} = (-eN) \frac{\rho}{3\varepsilon_0} x_p = -\frac{4\pi}{9\varepsilon_0} R^3 e^2 n^2 x_p \mathbf{e}_x$ (note the antiparallel attractive force is proportional to the displacement that it is similar to Hooke’s law). The work done on the electron cloud to shift it is $W_{el} = -\int_0^{x_p} F(x') \, dx' = \frac{1}{2} \left(\frac{4\pi}{9\varepsilon_0} R^3 e^2 n^2\right) x_p^2$. | 1.0 |

The spherical silver nanoparticle in an external constant electric field

|               | Inside the metallic particle in the steady state the electric field must be equal to 0. The induced field (from 2.2 or 2.3) compensates the external field: $E_0 + E_{\text{ind}} = 0$, so | 0.6 |

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\[ x_p = \frac{3\varepsilon_0}{\rho} E_0 = \frac{3\varepsilon_0}{en} E_0. \]
Charge displaced through the \( yz \)-plane is the total charge of electrons in the cylinder of radius \( R \) and height \( x_p \):

\[ -\Delta Q = -\rho \pi R^2 x_p = -\pi R^2 ne x_p. \]

## The equivalent capacitance and inductance of the silver nanoparticle

**2.5a** The electric energy \( W_{el} \) of a capacitor with capacitance \( C \) holding charges \( \pm \Delta Q \) is

\[ W_{el} = \frac{\Delta Q^2}{2C}. \]

The energy of such capacitor is equal to the work (see 2.3) done to separate the charges (see 2.4), thus

\[ C = \frac{\Delta Q^2}{2W_{el}} = \frac{9}{4} \varepsilon_0 \pi R = 6.26 \times 10^{-19} \text{ F}. \]

**0.7**

**2.5b** Equivalent scheme for a capacitor reads: \( \Delta Q = CV_0 \). Combining charge from (2.4) and capacitance from (2.5a) gives

\[ V_0 = \frac{\Delta Q}{C} = \frac{4}{3} R E_0. \]

**0.4**

**2.6a** The kinetic energy of the electron cloud is defined as the kinetic energy of one electron multiplied by the number of electrons in the cloud

\[ W_{kin} = \frac{1}{2} m_e v^2 N = \frac{1}{2} m_e v^2 \left( \frac{4}{3} \pi R^3 n \right). \]

The current \( I \) is the charge of electrons in the cylinder of area \( \pi R^2 \) and height \( \nu \Delta t \) divided by time \( \Delta t \) (or simply the time derivative of charge \( -\Delta Q \)), thus

\[ I = -e \nu \pi R^2. \]

**0.7**

**2.6b** The energy carried by current \( I \) in the equivalent circuit with inductance \( L \) is

\[ W = \frac{1}{2} LI^2 \]

is, in fact, the kinetic energy of electrons \( W_{kin} \). Taking the energy and current from (2.6a) gives

\[ L = \frac{4m_e}{3\pi R n e^2} = 2.57 \times 10^{-14} \text{ H}. \]

**0.5**

## The plasmon resonance of the silver nanoparticle

**2.7a** From the LC-circuit analogy we can directly derive

\[ \omega_p = (LC)^{-1/2} = \sqrt{\varepsilon_0/3e_m}. \]

Alternatively it is possible to use the harmonic law of motion in (2.3) and get the same result for the frequency.

**0.5**

**2.7b** \( \omega_p = 7.88 \times 10^{15} \text{ rad/s} \), for light with angular frequency \( \omega = \omega_p \) the wavelength is

\[ \lambda_p = \frac{2\pi c}{\omega_p} = 239 \text{ nm}. \]

**0.4**

## The silver nanoparticle illuminated with light at the plasmon frequency

**2.8a** The velocity of an electron \( v = \frac{dx}{dt} = -\omega x_0 \sin \omega t = v_0 \sin \omega t \). The time-averaged kinetic energy on the electron \( \langle W_k \rangle = \frac{m_e v^2}{2} = \frac{m_e}{2} \langle v^2 \rangle \). During time \( \tau \) each electron hits an ion one time. So the energy lost in the whole nanoparticle during time \( \tau \)

\[ W_{heat} = N \left( \frac{m_e v^2}{2} \right) = \frac{4}{3} \pi R^3 n \left( \frac{m_e}{2} v^2 \right). \]

Time-averaged Joule heating power

\[ P_{heat} = \frac{1}{\tau} W_{kin} = \frac{1}{2\tau} m_e \langle v^2 \rangle \left( \frac{4}{3} \pi R^3 n \right). \]

The expression for current is taken from (2.6a), squared and averaged

**1.0**
The average time between the collisions is \( \tau \gg 1/\omega_p \), so each electron oscillates many times before it collides with an ion. The oscillating current \( I = I_0 \sin \omega t = \pi R^2 ne \nu_0 \sin \omega t \) produces the heat in the resistance \( R_{\text{heat}} \) equal to \( P_{\text{heat}} = R_{\text{heat}} \langle I^2 \rangle \), that together with results from (2.8a) leads to \( R_{\text{heat}} = \frac{W_{\text{kin}}}{\pi \langle I^2 \rangle} = \frac{2m_e}{3\pi n e^2 R_t} = 2.46 \, \Omega \).

For equivalent scattering resistance \( R_{\text{scat}} = \frac{P_{\text{scat}}}{\langle I^2 \rangle} \) and for harmonic oscillations we can average the velocity squared over one period of oscillations, so \( \langle v^2 \rangle = \frac{1}{2} \omega_p^2 x_0^2 \). Together it yields \( R_{\text{scat}} = \frac{Q^2 x_0^3 \omega_0^2}{12\pi \varepsilon_0 c^3} \frac{16R^2}{9Q^2\langle v^2 \rangle} = \frac{8\omega_0^2 R^2}{27\pi \varepsilon_0 c^3} = 2.45 \, \Omega \).

Ohm’s law for a LCR serious circuit is \( I_0 = \frac{V_0}{\sqrt{(R_{\text{heat}} + R_{\text{scat}})^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \). At the resonance frequency time-averaged voltage squared is \( \langle V^2 \rangle = Z_0^2 \langle I^2 \rangle = (R_{\text{heat}} + R_{\text{scat}})^2 \langle I^2 \rangle \). And from (2.5b) \( \langle V^2 \rangle = \frac{1}{2} V_0^2 = \frac{8}{9} R^2 E_0^2 \), so Ohm’s law results in \( \langle I^2 \rangle = \frac{8R_{\text{heat}}^2}{9(R_{\text{heat}} + R_{\text{scat}})^2} E_0^2 \) and

\[
\left( R_{\text{scat}} \right) = \frac{8R_{\text{scat}}^2}{9(R_{\text{heat}} + R_{\text{scat}})^2} E_0^2 = \frac{R_{\text{scat}}}{R_{\text{heat}}} \left( P_{\text{heat}} \right).
\]

Starting with the electric field amplitude \( E_0 = \sqrt{2S/(\varepsilon_0 c)} = 27.4 \, \text{kV/m} \), we calculate \( P_{\text{heat}} = 6.82 \, \text{nW} \) and \( P_{\text{scat}} = 6.81 \, \text{nW} \).

**Steam generation by light**

Total number of nanoparticles in the vessel: \( N_{np} = h^2 a n_{np} = 7.3 \times 10^{11} \). Then the total time-averaged Joule heating power: \( P_{\text{st}} = N_{np} P_{\text{heat}} = 4.98 \, \text{kW} \). This power goes into the steam generation: \( P_{\text{st}} = \mu_{\text{st}} L_{\text{tot}} \), with \( L_{\text{tot}} = c_{\text{wa}} (T_{100} - T_{\text{wa}}) + L_{\text{wa}} + c_{\text{wa}} (T_{\text{st}} - T_{100}) = 2.62 \times 10^6 \, \text{J} \, \text{kg}^{-1} \). Thus the mass of steam produced in one second is: \( \mu_{\text{st}} = \frac{P_{\text{st}}}{L_{\text{tot}}} = 1.90 \times 10^{-3} \, \text{kg} \, \text{s}^{-1} \).

The power of light incident on the vessel \( P_{\text{tot}} = h^2 S = 0.01 \, \text{m}^2 \times 1 \, \text{MW} \, \text{m}^{-2} = 10.0 \, \text{kW} \), and the power directed for steam production by nanoparticles is given in 2.11a. Efficiency of the process is \( \eta = \frac{P_{\text{st}}}{P_{\text{tot}}} = \frac{4.98 \, \text{kW}}{10.0 \, \text{kW}} = 0.498 \).

**Total**

\( 12.0 \)