I. Solution

1.1 Let O be their centre of mass. Hence

\[ MR - mr = 0 \]

……………………… (1)

\[ m \omega_0^2 r = \frac{G M m}{(R + r)^2} \]

\[ M \omega_0^2 R = \frac{G M m}{(R + r)^2} \]

……………………… (2)

From Eq. (2), or using reduced mass, \( \omega_0^2 = \frac{G(M + m)}{(R + r)^3} \)

Hence, \( \omega_0^2 = \frac{G(M + m)}{(R + r)^3} = \frac{GM}{r(R + r)^2} = \frac{Gm}{R(R + r)^2} \). ………………… (3)
1.2 Since $\mu$ is infinitesimal, it has no gravitational influences on the motion of neither $M$ nor $m$. For $\mu$ to remain stationary relative to both $M$ and $m$ we must have:

$$\frac{GM\mu}{r_1^2}\cos\theta_1 + \frac{GM\mu}{r_2^2}\cos\theta_2 = \mu\omega^2_0\rho = \frac{G(M+m)\mu}{(R+r)^2}\rho$$ \hspace{1cm} \text{(4)}

$$\frac{GM\mu}{r_1^2}\sin\theta_1 = \frac{Gm\mu}{r_2^2}\sin\theta_2$$ \hspace{1cm} \text{(5)}

Substituting $\frac{GM}{r_1^2}$ from Eq. (5) into Eq. (4), and using the identity

$$\sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2 = \sin(\theta_1 + \theta_2),$$

we get

$$m\frac{\sin(\theta_1 + \theta_2)}{r_2^2} = \frac{(M+m)}{(R+r)^2}\rho \sin\theta_1$$ \hspace{1cm} \text{(6)}

The distances $r_2$ and $\rho$, the angles $\theta_1$ and $\theta_2$ are related by two Sine Rule equations

$$\frac{\sin\psi_1}{\rho} = \frac{\sin\theta_1}{R}$$

$$\frac{\sin\psi_1}{r_2} = \frac{\sin(\theta_1 + \theta_2)}{R+r}$$ \hspace{1cm} \text{(7)}

Substitute (7) into (6)

$$\frac{1}{r_2^3} = \frac{R}{(R+r)^2}\frac{(M+m)}{m}$$ \hspace{1cm} \text{(10)}

Since $\frac{m}{M+m} = \frac{R}{R+r}$, Eq. (10) gives

$$r_2 = R+r$$ \hspace{1cm} \text{(11)}

By substituting $\frac{Gm}{r_2^2}$ from Eq. (5) into Eq. (4), and repeat a similar procedure, we get

$$r_1 = R+r$$ \hspace{1cm} \text{(12)}

Alternatively,

$$\frac{r_1}{\sin(180^\circ - \phi)} = \frac{R}{\sin\theta_1} \quad \text{and} \quad \frac{r_2}{\sin\phi} = \frac{r}{\sin\theta_2}$$

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{R \times r_2}{r \times r_1} = \frac{m \times r_2}{M \times r_1}$$

Combining with Eq. (5) gives $r_1 = r_2$. 
Hence, it is an equilateral triangle with
\[ \psi_1 = 60^\circ \]
\[ \psi_2 = 60^\circ \] ................................. (13)

The distance \( \rho \) is calculated from the Cosine Rule.
\[ \rho^2 = r^2 + (R+r)^2 - 2r(R+r) \cos 60^\circ \]
\[ \rho = \sqrt{r^2 + rR + R^2} \] ................................. (14)

**Alternative Solution to 1.2**

Since \( \mu \) is infinitesimal, it has no gravitational influences on the motion of neither \( M \) nor \( m \). For \( \mu \) to remain stationary relative to both \( M \) and \( m \) we must have:

\[ \frac{GM \mu}{r_1^2} \cos \theta_1 + \frac{Gm \mu}{r_2^2} \cos \theta_2 = \mu \omega^2 \rho = \frac{G(M+m)\mu}{(R+r)^3} \rho \] ................................. (4)

\[ \frac{GM \mu}{r_1^2} \sin \theta_1 = \frac{Gm \mu}{r_2^2} \sin \theta_2 \] ................................. (5)

Note that
\[ \frac{r_1}{\sin(180^\circ - \phi)} = \frac{R}{\sin \theta_1} \]
\[ \frac{r_2}{\sin \phi} = \frac{r}{\sin \theta_2} \] (see figure)
\[ \frac{\sin \theta_1}{\sin \theta_2} = \frac{R \times r_2}{r \times r_1} = \frac{m}{M} \times \frac{r_2}{r_1} \] ................................. (6)

Equations (5) and (6):
\[ r_1 = r_2 \] ................................. (7)
\[ \frac{\sin \theta_1}{\sin \theta_2} = \frac{m}{M} \] ................................. (8)
\[ \psi_1 = \psi_2 \] ................................. (9)

The equation (4) then becomes:
\[ M \cos \theta_1 + m \cos \theta_2 = \frac{(M+m)}{(R+r)^3} r_1^2 \rho \] ................................. (10)

Equations (8) and (10):
\[ \sin (\theta_1 + \theta_2) = \frac{M+m}{M} \frac{r_2^2 \rho}{(R+r)^3} \sin \theta_2 \] ................................. (11)

Note that from figure,
\[ \frac{\rho}{\sin \psi_2} = \frac{r}{\sin \theta_2} \] ................................. (12)
Equations (11) and (12): \[ \sin(\theta_1 + \theta_2) = \frac{M + m}{M} \frac{r_1^2 r}{(R+r)^2} \sin \psi_2 \] ............................ (13)

Also from figure,
\[ (R + r)^2 = r_1^2 - 2r_1 r_2 \cos(\theta_1 + \theta_2) + r_2^2 = 2r_1^2 \left[ 1 - \cos(\theta_1 + \theta_2) \right] \] ............................ (14)

Equations (13) and (14): \[ \sin(\theta_1 + \theta_2) = \frac{\sin \psi_2}{2 \left[ 1 - \cos(\theta_1 + \theta_2) \right]} \] ............................ (15)

\[ \theta_1 + \theta_2 = 180^\circ - \psi_1 - \psi_2 = 180^\circ - 2\psi_2 \] (see figure)
\[ \therefore \cos \psi_2 = \frac{1}{2}, \psi_2 = 60^\circ, \psi_1 = 60^\circ \]

Hence \( M \) and \( m \) from an equilateral triangle of sides \( (R + r) \)
Distance \( \mu \) to \( M \) is \( R + r \)
Distance \( \mu \) to \( m \) is \( R + r \)
Distance \( \mu \) to \( O \) is \[ \rho = \sqrt{\left(\frac{R + r}{2} - R\right)^2 + \left(\frac{R + r}{2}\right)^2} = \sqrt{R^2 + Rr + r^2} \]

1.3 The energy of the mass \( \mu \) is given by

\[ E = -\frac{GM \mu}{r_1} - \frac{Gm \mu}{r_2} + \frac{1}{2} \mu \left( \frac{d \rho}{dt} \right)^2 + \rho^2 \omega^2 \] ............................ (15)

Since the perturbation is in the radial direction, angular momentum is conserved \((r_1 = r_2 = R \text{ and } m = M)\),
\[ E = -\frac{2GM \mu}{R} + \frac{1}{2} \mu \left( \frac{d \rho}{dt} \right)^2 + \frac{\rho_0^4 \omega_0^2}{\rho^2} \] ............................ (16)

Since the energy is conserved,
\[ \frac{dE}{dt} = 0 \]
\[ \frac{dE}{dt} = \frac{2GM \mu}{R^2} \frac{d\cal R}{dt} + \mu \frac{d \rho}{dt} \frac{d^2 \rho}{dt^2} - \mu \frac{\rho_0^4 \omega_0^2}{\rho^2} \frac{d \rho}{dt} = 0 \] ............................ (17)

\[ \frac{d\cal R}{dt} = \frac{d \rho}{dt} \frac{\cal R}{\rho} = \frac{d \rho}{dt} \cal R \] ............................ (18)
\[ \frac{dE}{dt} = \frac{2GM \mu}{\cal R^3} \rho \frac{d \rho}{dt} + \mu \frac{d \rho}{dt} \frac{d^2 \rho}{dt^2} - \mu \frac{\rho_0^4 \omega_0^2}{\rho^2} \frac{d \rho}{dt} = 0 \] ............................ (19)
Since \( \frac{d\rho}{dt} \neq 0 \), we have

\[
\frac{2GM}{R^3} - \frac{d^2\rho}{dt^2} - \frac{\rho_0^4 \omega_0^2}{\rho^3} = 0 \quad \text{or} \quad \frac{d^2\rho}{dt^2} = -\frac{2GM}{R^3} \rho + \frac{\rho_0^4 \omega_0^2}{\rho^3}.
\]

..........................(20)

The perturbation from \( R_0 \) and \( \rho_0 \) gives \( R = R_0 \left(1 + \frac{\Delta R}{R_0}\right) \) and \( \rho = \rho_0 \left(1 + \frac{\Delta \rho}{\rho_0}\right) \).

Then

\[
\frac{d^2\rho}{dt^2} = d^2(r_0 + \Delta r) = - \frac{2GM}{R_0^3} \rho_0 \left(1 + \frac{\Delta \rho}{\rho_0}\right) \left(1 + \frac{\Delta R}{R_0}\right) + \rho_0 \omega_0^2 \left(1 - \frac{3\Delta \rho}{\rho_0}\right).
\]

..........................(21)

Using binomial expansion \((1 + \varepsilon)^n \approx 1 + n\varepsilon\),

\[
\frac{d^3\Delta \rho}{dt^2} = - \frac{2GM}{R_0^3} \rho_0 \left(1 + \frac{\Delta \rho}{\rho_0}\right) \left(1 - \frac{3\Delta R}{R_0}\right) + \rho_0 \omega_0^2 \left(1 - \frac{3\Delta \rho}{\rho_0}\right).
\]

..........................(22)

Using \( \Delta \rho = \frac{R}{\rho} \Delta R \),

\[
\frac{d^3\Delta \rho}{dt^2} = - \frac{2GM}{R_0^3} \rho_0 \left(1 + \frac{\Delta \rho}{\rho_0} - \frac{3\rho_0 \Delta \rho}{R_0^2}\right) + \rho_0 \omega_0^2 \left(1 - \frac{3\Delta \rho}{\rho_0}\right).
\]

..........................(23)

Since \( \omega_0^2 = \frac{2GM}{R_0^3} \),

\[
\frac{d^3\Delta \rho}{dt^2} = - \omega_0^2 \rho_0 \left(1 + \frac{\Delta \rho}{\rho_0} - \frac{3\rho_0 \Delta \rho}{R_0^2}\right) + \omega_0^2 \rho_0 \left(1 - \frac{3\Delta \rho}{\rho_0}\right)
\]

..........................(24)

\[
\frac{d^3\Delta \rho}{dt^2} = - \omega_0^2 \rho_0 \left(4 - \frac{3\rho_0 \Delta \rho}{R_0^2}\right)
\]

..........................(25)

\[
\frac{d^3\Delta \rho}{dt^2} = - \omega_0^2 \Delta \rho \left(4 - \frac{9}{4}\right) = - \frac{7}{4} \omega_0^2 \Delta \rho.
\]

..........................(27)
Angular frequency of oscillation is $\frac{\sqrt{7}}{2} a_0$.

Alternative solution:

$M = m$ gives $R = r$ and $\omega_0^2 = \frac{G(M + M)}{(R + R)^3} = \frac{GM}{4R^3}$. The unperturbed radial distance of $\mu$ is $\sqrt{3}R$, so the perturbed radial distance can be represented by $\sqrt{3}R + \zeta$ where $\zeta << \sqrt{3}R$ as shown in the following figure.

Using Newton’s 2nd law, $-\frac{2GM\mu}{\{R^2 + (\sqrt{3}R + \zeta)^2\}^{3/2}}(\sqrt{3}R + \zeta) = \mu \frac{d^2}{dt^2}(\sqrt{3}R + \zeta) - \mu\omega^2(\sqrt{3}R + \zeta)$.

(1)

The conservation of angular momentum gives $\mu\omega_0(\sqrt{3}R)^2 = \mu\omega(\sqrt{3}R + \zeta)^2$.

(2)

Manipulate (1) and (2) algebraically, applying $\zeta^2 \approx 0$ and binomial approximation.

$$-\frac{2GM}{\{R^2 + (\sqrt{3}R + \zeta)^2\}^{3/2}}(\sqrt{3}R + \zeta) = \frac{d^2}{dt^2}(\sqrt{3}R + \zeta) - \frac{\omega_0^2 \sqrt{3}R}{(1 + \zeta / \sqrt{3}R)^3}$$

$$-\frac{2GM}{\{4R^2 + 2\sqrt{3}R\zeta\}^{3/2}}(\sqrt{3}R + \zeta) \approx \frac{d^2}{dt^2}(\sqrt{3}R + \zeta) - \frac{\omega_0^2 \sqrt{3}R}{(1 + \zeta / \sqrt{3}R)^3}$$

$$-\frac{GM\sqrt{3}R}{4R^3}(1 + \zeta / \sqrt{3}R) = \frac{d^2}{dt^2}(\sqrt{3}R + \zeta) - \frac{\omega_0^2 \sqrt{3}R}{(1 + \zeta / \sqrt{3}R)^3}$$

$$-\omega_0^2 \sqrt{3}R \left(1 - \frac{3\sqrt{3}\zeta}{4R}\right) \left(1 + \frac{\zeta}{\sqrt{3}R}\right) \approx \frac{d^2}{dt^2} - \omega_0^2 \sqrt{3}R \left(1 - \frac{3\zeta}{\sqrt{3}R}\right)$$

$$\frac{d^2}{dt^2} \zeta = -\left(\frac{7}{4} \omega_0^2\right) \zeta$$

1.4 Relative velocity

Let $v$ = speed of each spacecraft as it moves in circle around the centre O.
The relative velocities are denoted by the subscripts A, B and C.
For example, $v_{BA}$ is the velocity of B as observed by A.

The period of circular motion is 1 year $T = 365 \times 24 \times 60 \times 60$ s. \hspace{2cm} \cdots (28)
The angular frequency $\omega = \frac{2\pi}{T}$
The speed $v = \omega \frac{L}{2 \cos 30^\circ} = 575$ m/s \hspace{2cm} \cdots (29)
The speed is much less than the speed light $\rightarrow$ Galilean transformation.

In Cartesian coordinates, the velocities of B and C (as observed by O) are

For B, $v_B = \cos 60^\circ \hat{i} - \sin 60^\circ \hat{j}$

For C, $v_C = \cos 60^\circ \hat{i} + \sin 60^\circ \hat{j}$

Hence $v_{BC} = -2\sin 60^\circ \hat{j} = -\sqrt{3}v \hat{j}$

The speed of B as observed by C is $\sqrt{3}v \approx 996$ m/s

Notice that the relative velocities for each pair are anti-parallel.

**Alternative solution for 1.4**

One can obtain $v_{BC}$ by considering the rotation about the axis at one of the spacecrafts.

$v_{BC} = \frac{\omega L}{2\pi} = \frac{2\pi}{365 \times 24 \times 60 \times 60 \text{ s}} (5 \times 10^6 \text{ km}) \approx 996$ m/s