1. The only neutrons that will survive absorption at A are those that cannot cross $H$. Their turning points will be below $H$. So that, for a neutron entering to the cavity at height $z$ with vertical velocity $v_z$, conservation of energy implies

$$\frac{1}{2} M v_z^2 + M g z \leq M g H \quad \Rightarrow \quad -\sqrt{2g(H-z)} \leq v_z(z) \leq \sqrt{2g(H-z)}$$

2. The cavity should be long enough to ensure the absorption of all neutrons with velocities outside the allowed range. Therefore, neutrons have to reach its maximum height at least once within the cavity. The longest required length corresponds to neutrons that enter at $z = H$ with $v_z = 0$ (see the figure). Calling $t_f$ to their time of fall

$$L_c = v_x 2 t_f \quad H = \frac{1}{2} g t_f^2$$

$$\Rightarrow \quad L_c = 2 v_x \sqrt{\frac{2H}{g}} \quad L_c = 6.4 \text{ cm}$$

3. The rate of transmitted neutrons entering at a given height $z$, per unit height, is proportional to the range of allowed velocities at that height, $\rho$ being the proportionality constant

$$\frac{dN_c(z)}{dz} = \rho [v_{z,max}(z) - v_{z,min}(z)] = 2 \rho \sqrt{2g(H-z)}$$

The total number of transmitted neutrons is obtained by adding the neutrons entering at all possible heights. Calling $y = z / H$

$$N_c(H) = \int_0^H \frac{dN_c(z)}{dz} = \int_0^H 2 \rho \sqrt{2g(H-z)} dz = 2 \rho \sqrt{2g} H^{3/2} \int_0^1 (1 - y)^{1/2} dy = 2 \rho \sqrt{2g} H^{3/2} \left[ -\frac{2}{3} (1 - y)^{3/2} \right]_0^1$$

$$\Rightarrow \quad N_c(H) = \frac{4}{3} \rho \sqrt{2g} H^{3/2}$$

4. For a neutron falling from a height $H$, the action over a bouncing cycle is twice the action during the fall or the ascent

$$S = \int_0^H p_z dz = 2M \sqrt{2g} H^{3/2} \int_0^1 (1 - y)^{1/2} dy = \frac{4}{3} M \sqrt{2g} H^{3/2}$$

Using the BS quantization condition

$$S = \frac{4}{3} M \sqrt{2g} H^{3/2} = n \hbar$$

$$\Rightarrow \quad H_n = \left( \frac{9 \hbar^2}{32 M^2 g} \right)^{1/3} n^{2/3}$$

The corresponding energy levels (associated to the vertical motion) are

$$E_n = M g H_n \quad \Rightarrow \quad E_n = \left( \frac{9M g^2 \hbar^2}{32} \right)^{1/3} n^{2/3}$$
Numerical values for the first level:

\[ H_1 = \left( \frac{9h^2}{32M^2g} \right)^{1/3} = 1.65 \times 10^{-5} \text{ m} \quad H_1 = 16.5 \text{ µm} \]

\[ E_1 = M g H_1 = 2.71 \times 10^{-31} \text{ J} = 1.69 \times 10^{-12} \text{ eV} \quad E_1 = 1.69 \text{ peV} \]

Note that \( H_1 \) is of the same order than the given cavity height, \( H = 50 \text{ µm} \). This opens up the possibility for observing the spatial quantization when varying \( H \).

5. The uncertainty principle says that the minimum time \( \Delta t \) and the minimum energy \( \Delta E \) satisfy the relation \( \Delta E \Delta t \geq \hbar \).

During this time, the neutrons move to the right a distance

\[ \Delta x = v_x \Delta t \geq v_x \frac{\hbar}{\Delta E} \]

Now, the minimum neutron energy allowed in the cavity is \( E_1 \), so that \( \Delta E \approx E_1 \). Therefore, an estimation of the minimum time and the minimum length required is

\[ t_q \approx \frac{\hbar}{E_1} = 0.4 \cdot 10^{-3} \text{ s} = 0.4 \text{ ms} \]

\[ L_q \approx v_x \frac{\hbar}{E_1} = 4 \cdot 10^{-3} \text{ m} = 4 \text{ mm} \]
<table>
<thead>
<tr>
<th>Question</th>
<th>Basic formulas used</th>
<th>Analytical results</th>
<th>Numerical results</th>
<th>Marking guideline</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \frac{1}{2} M v_z^2 + M g z \leq M g H )</td>
<td>( -\sqrt{2g(H-z)} \leq v_z(z) \leq \sqrt{2g(H-z)} )</td>
<td></td>
<td>1.5</td>
</tr>
<tr>
<td>2</td>
<td>( L_c = v_x \sqrt{\frac{2H}{g}} )</td>
<td>( L_c = \frac{1}{2} g t_f^2 )</td>
<td>( L_c = 6.4 ) cm</td>
<td>1.3 + 0.2</td>
</tr>
<tr>
<td>3</td>
<td>( \frac{dN_c}{dz} = \rho [v_{z,\text{max}} - v_{z,\text{min}}] )</td>
<td>( N_c(H) = \frac{4}{3} \rho \sqrt{2g} H^{3/2} )</td>
<td></td>
<td>2.5</td>
</tr>
<tr>
<td>4</td>
<td>( S = 2 \int_0^H p_z dz = nh )</td>
<td>( H_n = \left( \frac{9h^2}{32 M^2 g} \right)^{1/3} n^{2/3} )</td>
<td>( H_1 = 16.5 \mu m )</td>
<td>1.6 + 0.2</td>
</tr>
<tr>
<td>5</td>
<td>( \Delta E \Delta t \geq \hbar )</td>
<td>( t_q \approx \frac{\hbar}{E_1} )</td>
<td>( t_q \approx 0.4 ) ms</td>
<td>1.3 + 0.2</td>
</tr>
<tr>
<td></td>
<td>( \Delta E \approx E_1 )</td>
<td>( L_q \approx \frac{\hbar}{E_1} )</td>
<td>( L_q \approx 4 ) mm</td>
<td>0.3 + 0.2</td>
</tr>
</tbody>
</table>