Th 2  ABSOLUTE MEASUREMENTS OF ELECTRICAL QUANTITIES

SOLUTION

1. After some time $t$, the normal to the coil plane makes an angle $\omega t$ with the magnetic field $\vec{B}_0 = B_0 \hat{i}$. Then, the magnetic flux through the coil is

$$\phi = N \vec{B}_0 \cdot \vec{S}$$

where the vector surface $\vec{S}$ is given by $\vec{S} = \pi a^2 \left( \cos \omega t \hat{i} + \sin \omega t \hat{j} \right)$

Therefore $\phi = N \pi a^2 B_0 \cos \omega t$

The induced electromotive force is

$$\varepsilon = -\frac{d\phi}{dt}$$

$$\Rightarrow \varepsilon = N \pi a^2 B_0 \omega \sin \omega t$$

The instantaneous power is $P = \varepsilon^2 / R$, therefore

$$\langle P \rangle = \frac{\left( N \pi a^2 B_0 \omega \right)^2}{2R}$$

where we used $\langle \sin^2 \omega t \rangle = \frac{1}{T} \int_0^T \sin^2 \omega t \, dt = \frac{1}{2}$

2. The total field at the center the coil at the instant $t$ is

$$\vec{B}_t = \vec{B}_0 + \vec{B}_i$$

where $\vec{B}_i$ is the magnetic field due to the induced current $\vec{B}_i = B_i \left( \cos \omega t \hat{i} + \sin \omega t \hat{j} \right)$

with $B_i = \frac{\mu_0 N I}{2a}$ and $I = \varepsilon / R$

Therefore $B_i = \frac{\mu_0 N^2 \pi a B_0 \omega}{2R} \sin \omega t$

The mean values of its components are

$$\langle B_{ix} \rangle = \frac{\mu_0 N^2 \pi a B_0 \omega}{2R} \langle \sin \omega t \cos \omega t \rangle = 0$$

$$\langle B_{iy} \rangle = \frac{\mu_0 N^2 \pi a B_0 \omega}{2R} \langle \sin^2 \omega t \rangle = \frac{\mu_0 N^2 \pi a B_0 \omega}{4R}$$

And the mean value of the total magnetic field is

$$\langle \vec{B}_i \rangle = B_0 \hat{i} + \frac{\mu_0 N^2 \pi a B_0 \omega}{4R} \hat{j}$$

The needle orients along the mean field, therefore

$$\tan \theta = \frac{\mu_0 N^2 \pi a \omega}{4R}$$
Finally, the resistance of the coil measured by this procedure, in terms of $\theta$, is

$$R = \frac{\mu_0 N^2 \pi a \omega}{4 \tan \theta}$$

3. The force on a unit positive charge in a disk is radial and its modulus is

$$|\vec{v} \times \vec{B}| = vB = \omega r B$$

where $B$ is the magnetic field at the center of the coil

$$B = N \frac{\mu_0 I}{2a}$$

Then, the electromotive force (e.m.f.) induced on each disk by the magnetic field $B$ is

$$\varepsilon_D = \varepsilon_{D'} = B\omega \int_0^b r \, dr = \frac{1}{2} B \omega b^2$$

Finally, the induced e.m.f. between 1 and 4 is $\varepsilon = \varepsilon_D + \varepsilon_D'$

$$\varepsilon = N \frac{\mu_0 b^2 \omega I}{2a}$$

4. When the reading of G vanishes, $I_G = 0$ and Kirchoff laws give an immediate answer. Then we have

$$\varepsilon = I_R \Rightarrow R = N \frac{\mu_0 b^2 \omega}{2a}$$

5. The force per unit length $f$ between two indefinite parallel straight wires separated by a distance $h$ is

$$f = \frac{\mu_0 I_1 I_2}{2\pi h}$$

for $I_1 = I_2 = I$ and length $2\pi a$, the force $F$ induced on $C_2$ by the neighbor coils $C_1$ is

$$F = \frac{\mu_0 a^2 I^2}{h}$$

6. In equilibrium

$$mgx = 4Fd$$

Then

$$mgx = \frac{4\mu_0 ad}{h} f^2$$

so that

$$f = \sqrt{\frac{mgx}{4\mu_0 ad}}$$
7. The balance comes back towards the equilibrium position for a little angular deviation $\delta \varphi$ if the gravity torques with respect to the fulcrum O are greater than the magnetic torques.

$$Mgl \sin \delta \varphi + mgx \cos \delta \varphi > 2\mu_0 aI^2 \left( \frac{1}{h - \delta z} + \frac{1}{h + \delta z} \right) \cos \delta \varphi$$

![Diagram of a balance with gravity and magnetic torques]

Therefore, using the suggested approximation

$$Mgl \sin \delta \varphi + mgx \cos \delta \varphi > \frac{4\mu_0 adI^2}{h} \left( 1 + \frac{\delta z^2}{h^2} \right) \cos \delta \varphi$$

Taking into account the equilibrium condition (1), one obtains

$$Mgl \sin \delta \varphi > mgx \frac{\delta z^2}{h^2} \cos \delta \varphi$$

Finally, for $\tan \delta \varphi \approx \sin \delta \varphi = \frac{\delta z}{d}$

$$\delta z < \frac{Mlh^2}{mxd} \quad \Rightarrow \quad \delta z_{\text{max}} = \frac{Mlh^2}{mxd}$$
### Th 2  ANSWER SHEET

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<th>Question</th>
<th>Basic formulas and ideas used</th>
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| 1        | \( \Phi = N \mathbf{\vec{B}}_0 \cdot \mathbf{\vec{S}} \)  
\( E = \frac{d\Phi}{dt} \)  
\( P = \frac{E^2}{R} \) | \( E = N\pi a^2 B_0 \omega \sin \omega t \)  
\( \langle P \rangle = \left( \frac{N\pi a^2 B_0 \omega}{2R} \right)^2 \) | 0.5  
1.0 |
| 2        | \( \mathbf{\vec{B}} = \mathbf{\vec{B}}_0 + \mathbf{\vec{B}}_i \)  
\( B_i = \frac{\mu_0 N I}{2a} \)  
\( \tan \theta = \frac{\langle B_y \rangle}{\langle B_z \rangle} \) | \( R = \frac{\mu_0 N^2 \pi a \omega}{4 \tan \theta} \) | 2.0 |
| 3        | \( \mathbf{\vec{E}} = \mathbf{\vec{v}} \times \mathbf{\vec{B}} \)  
\( \mathbf{\vec{v}} = \omega \mathbf{\vec{r}} \)  
\( B = N \frac{\mu_0 I}{2a} \)  
\( \mathbf{\vec{E}} = \int_0^b \mathbf{\vec{E}} \, d\mathbf{\vec{r}} \) | \( E = N \frac{\mu_0 b^2 \omega I}{2a} \) | 2.0 |
| 4        | \( E = RL \) | \( R = N \frac{\mu_0 b^2 \omega}{2a} \) | 0.5 |
| 5        | \( f = \frac{\mu_0 I I'}{2\pi h} \) | \( F = \frac{\mu_0 a}{h} I^2 \) | 1.0 |
| 6        | \( mgx = 4Fd \) | \( I = \left( \frac{mgx}{4\mu_0 ad} \right)^{1/2} \) | 1.0 |
| 7        | \( \Gamma_{grav} > \Gamma_{mag} \) | \( \delta z_{max} = \frac{Ml^2}{mx^2} \) | 2.0 |