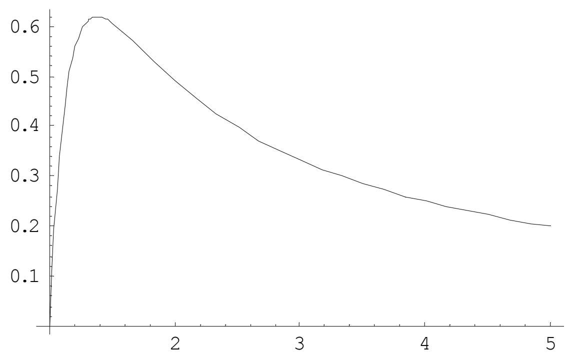
Theoretical Question 2: *Rising Balloon*

1. Answers

(a) $F_B = M_A ng \dfrac{P}{P + \Delta P}$

(b) $\gamma = \dfrac{\rho_0 z_0 g}{P_0} = 5.5$

(c) $\Delta P = \dfrac{4\kappa RT}{r_0}\left(\dfrac{1}{\lambda} - \dfrac{1}{\lambda^7}\right)$



(d) a =0.110

(e) z_f =11 km, λ_f =2.1.

2. Solutions

[Part A]

(a) [1.5 points]

Using the ideal gas equation of state, the volume of the helium gas of n moles at pressure $P+\Delta P$ and temperature T is

$$V = nRT/(P+\Delta P) \tag{a1}$$

while the volume of n' moles of air gas at pressure P and temperature T is

$$V = n'RT/P. \tag{a2}$$

Thus the balloon displaces $n' = n\dfrac{P}{P+\Delta P}$ moles of air whose weight is $M_A n' g$.

This displaced air weight is the buoyant force, i.e.,

$$F_B = M_A n g \frac{P}{P+\Delta P}. \tag{a3}$$

(Partial credits for subtracting the gas weight.)

(b) [2 points]

The pressure difference arising from a height difference of z is $-\rho g z$ when the air density ρ is a constant. When it varies as a function of the height, we have

$$\frac{dP}{dz} = -\rho g = -\frac{\rho_0 T_0}{P_0}\frac{P}{T} g \tag{b1}$$

where the ideal gas law $\rho T/P =$ constant is used. Inserting Eq. (2.1) and $T/T_0 = 1 - z/z_0$ on both sides of Eq. (b1), and comparing the two, one gets

$$\gamma = \frac{\rho_0 z_0 g}{P_0} = \frac{1.16\times 4.9\times 10^4 \times 9.8}{1.01\times 10^5} = 5.52. \tag{b2}$$

The required numerical value is 5.5.

[Part B]

(c) [2 points]

The work needed to increase the radius from r to $r+dr$ under the pressure difference ΔP is

$$dW = 4\pi r^2 \Delta P dr, \tag{c1}$$

while the increase of the elastic energy for the same change of r is

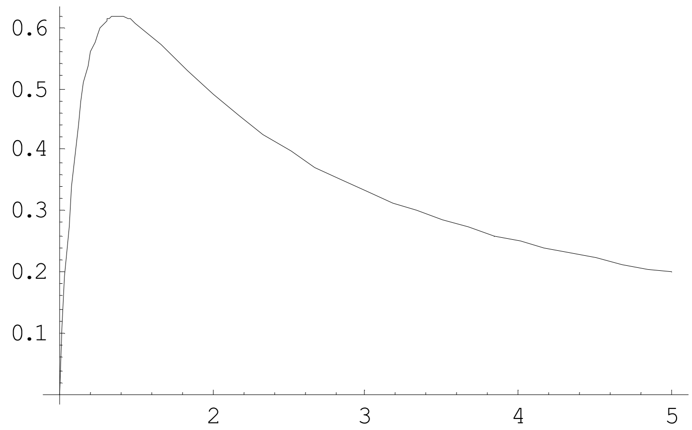
$$dW = \left(\frac{dU}{dr}\right)dr = 4\pi\kappa RT(4r - 4\frac{r_0^{\,6}}{r^5})dr\,. \tag{c2}$$

Equating the two expressions of dW, one gets

$$\Delta P = 4\kappa\, RT(\frac{1}{r} - \frac{r_0^{\,6}}{r^7}) \;=\; \frac{4\kappa RT}{r_0}\left(\frac{1}{\lambda} - \frac{1}{\lambda^7}\right). \tag{c3}$$

This is the required answer.

The graph as a function of λ (>1) increases sharply initially, has a maximum at $\lambda = 7^{1/6}$ =1.38, and decreases as λ^{-1} for large λ. The plot of $\Delta P/(4\kappa RT/r_0)$ is given below.



(d) [1.5 points]

From the ideal gas law,

$$P_0 V_0 = n_0 RT_0 \tag{d1}$$

where V_0 is the unstretched volume.

At volume $V = \lambda^3 V_0$ containing n moles, the ideal gas law applied to the gas inside at $T = T_0$ gives the inside pressure P_{in} as

$$P_{\text{in}} = nRT_0 / V = \frac{n}{n_0\lambda^3}P_0 \quad . \tag{d2}$$

On the other hand, the result of (c) at $T = T_0$ gives

$$P_{\text{in}} = P_0 + \Delta P = P_0 + \frac{4\kappa RT_0}{r_0}(\frac{1}{\lambda} - \frac{1}{\lambda^7}) = (1 + a(\frac{1}{\lambda} - \frac{1}{\lambda^7}))P_0\,. \tag{d3}$$

Equating (d2) and (d3) to solve for a,

$$a = \frac{n/(n_0\lambda^3)-1}{\lambda^{-1}-\lambda^{-7}}. \tag{d5}$$

Inserting n/n_0=3.6 and λ=1.5 here, a=0.110.

[Part C]

(e) [3 points]

The buoyant force derived in problem (a) should balance the total mass of M_{T}=1.12 kg. Thus, from Eq. (a3), at the weight balance,

$$\frac{P}{P+\Delta P} = \frac{M_{\mathrm{T}}}{M_A n}. \tag{e1}$$

On the other hand, applying again the ideal gas law to the helium gas inside of volume $V = \frac{4}{3}\pi r^3 = \lambda^3 \frac{4}{3}\pi r_0{}^3 = \lambda^3 V_0$, for arbitrary ambient P and T, one has

$$(P+\Delta P)\lambda^3 = \frac{nRT}{V_0} = P_0\frac{T}{T_0}\frac{n}{n_0} \tag{e2}$$

for n moles of helium. Eqs. (c3), (e1), and (e2) determine the three unknowns P, ΔP, and λ as a function of T and other parameters. Using Eq. (e2) in Eq. (e1), one has an alternative condition for the weight balance as

$$\frac{P}{P_0}\frac{T_0}{T}\lambda^3 = \frac{M_{\mathrm{T}}}{M_A n_0}\quad . \tag{e3}$$

Next using (c3) for ΔP in (e2), one has

$$P\lambda^3 + \frac{4\kappa RT}{r_0}\lambda^2(1-\lambda^{-6}) = P_0\frac{T}{T_0}\frac{n}{n_0}$$

or, rearranging it,

$$\frac{P}{P_0}\frac{T_0}{T}\lambda^3 = \frac{n}{n_0} - a\lambda^2(1-\lambda^{-6}), \tag{e4}$$

where the definition of a has been used again.

Equating the right hand sides of Eqs. (e3) and (e4), one has the equation for λ as

$$\lambda^2(1-\lambda^{-6}) = \frac{1}{an_0}\left(n - \frac{M_{\mathrm{T}}}{M_A}\right) = 4.54. \tag{e5}$$

The solution for λ can be obtained by

$$\lambda^2 \approx 4.54/(1-4.54^{-3}) \approx 4.54: \quad \lambda_f \cong 2.13. \tag{e6}$$

To find the height, replace $(P/P_0)/(T/T_0)$ on the left hand side of Eq. (e3) as a function of the height given in (b) as

$$\frac{P}{P_0}\frac{T_0}{T}\lambda^3 = (1 - z_f / z_0)^{\gamma-1}\lambda_f^3 = \frac{M_{\mathrm{T}}}{M_A n_0} = 3.10\ . \tag{e7}$$

Solution of Eq. (e7) for z_f with λ_f=2.13 and $\gamma-1$=4.5 is

$$z_f = 49 \times \left(1 - (3.10/2.13^3)^{1/4.5}\right) = 10.9\ \text{(km)}. \tag{e8}$$

The required answers are $\lambda_f = 2.1$, and $z_f = 11$ km.

3. Mark Distribution

No.	Total Pt.	Partial Pt.	Contents
(a)	1.5	0.5	Archimedes' principle
		0.5	Ideal gas law applied correctly
		0.5	Correct answer (partial credits 0.3 for subtracting He weight)
(b)	2.0	0.8	Relation of pressure difference to air density
		0.5	Application of ideal gas law to convert the density into pressure
		0.5	Correct formula for γ
		0.2	Correct number in answer
(c)	2.0	0.7	Relation of mechanical work to elastic energy change
		0.3	Relation of pressure to force
		0.5	Correct answer in formula
		0.5	Correct sketch of the curve
(d)	1.5	0.3	Use of ideal gas law for the increased pressure inside
		0.4	Expression of inside pressure in terms of a at the given conditions
		0.5	Formula or correct expression for a
		0.3	Correct answer
(e)	3.0	0.3	Use of force balance as one condition to determine unknowns
		0.3	Ideal gas law applied to the gas as an independent condition to determine unknowns
		0.5	The condition to determine λ_f numerically
		0.7	Correct answer for λ_f
		0.5	The relation of z_f versus λ_f
		0.7	Correct answer for z_f
Total	10		