Solutions

PART-A Product of the mass and the position of the ball ($m \times l$) (4.0 points)

1. Suggest and justify, by using equations, a method allowing to obtain $m \times l$. (2.0 points)

$$m \times l = (M + m) \times l_{cm}$$

(Explanation) The lever rule is applied to the Mechanical "Black Box", shown in Fig. A-1, once the position of the center of mass of the whole system is found.

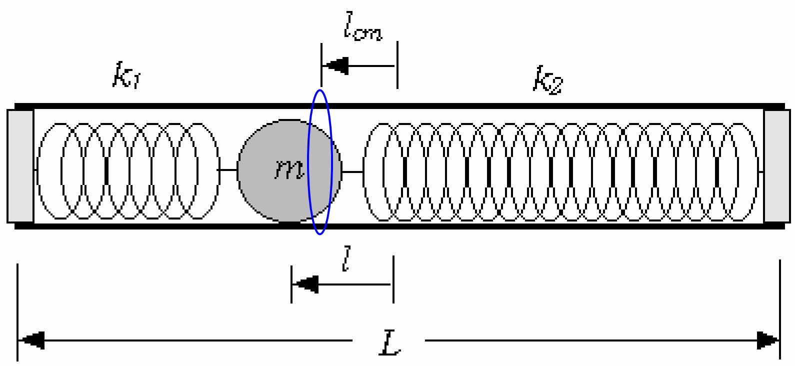

Fig. A-1 Experimental setup

2. Experimentally determine the value of $m \times l$. (2.0 points)

$$m \times l = 2.96 \times 10^{-3}\,\text{kg·m}$$

(Explanation) The measured quantities are

$$M + m = (1.411 \pm 0.0005) \times 10^{-1}\,\text{kg}$$

and

$$l_{cm} = (2.1 \pm 0.06) \times 10^{-2}\,\text{m} \quad \text{or} \quad 21 \pm 0.6\ \text{mm}.$$

Therefore

$$\begin{aligned} m \times l &= (M + m) \times l_{cm} \\ &= (1.411 \pm 0.0005) \times 10^{-1}\,\text{kg} \times (2.1 \pm 0.06) \times 10^{-2}\,\text{m} \\ &= (2.96 \pm 0.08) \times 10^{-3}\,\text{kg·m} \end{aligned}$$

PART-B The mass *m* of the ball (10.0 points)

1. Measure v for various values of h. Plot the data on a graph paper in a form that is suitable to find the value of m. Identify the slow rotation region and the fast rotation region on the graph. (4.0 points)

2. Show from your measurements that $h = Cv^2$ in the slow rotation region, and $h = Av^2+B$ in the fast rotation region. (1.0 points)

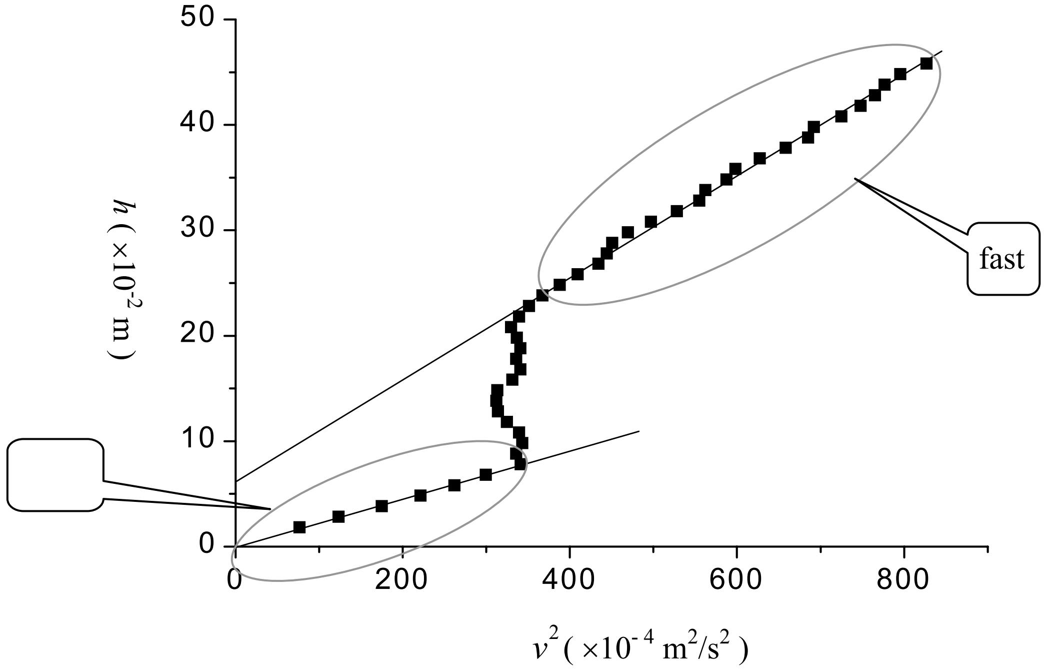

Fig. B-1 Experimental data

(Explanation) The measured data are

	h_1 ($\times10^{-2}$ m) a)	Δt (ms)	h ($\times10^{-2}$ m) b)	v ($\times10^{-2}$ m/s) c)	v^2 ($\times10^{-4}$ m²/s²)
1	25.5±0.1	269.4±0.05	1.8±0.1	8.75±0.02	76.6±0.2
2	26.5±0.1	235.7±0.05	2.8±0.1	11.12±0.02	123.7±0.3
3	27.5±0.1	197.9±0.05	3.8±0.1	13.24±0.03	175.3±0.6
4	28.5±0.1	176.0±0.05	4.8±0.1	14.89±0.03	221.7±0.6
5	29.5±0.1	161.8±0.05	5.8±0.1	16.19±0.03	262.1±0.7
6	30.5±0.1	151.4±0.05	6.8±0.1	17.31±0.03	299.6±0.7
7	31.5±0.1	141.8±0.05	7.8±0.1	18.48±0.04	342±1
8	32.5±0.1	142.9±0.05	8.8±0.1	18.33±0.04	336±1

9	33.5±0.1	141.4±0.05	9.8±0.1	18.53±0.04	343±1
10	34.5±0.1	142.2±0.05	10.8±0.1	18.42±0.04	339±1
11	35.5±0.1	145.4±0.05	11.8±0.1	18.02±0.04	325±1
12	36.5±0.1	147.8±0.05	12.8±0.1	17.73±0.04	314±1
13	37.5±0.1	148.3±0.05	13.8±0.1	17.67±0.04	312±1
14	38.5±0.1	148.0±0.05	14.8±0.1	17.70±0.04	313±1
15	39.5±0.1	143.9±0.05	15.8±0.1	18.21±0.04	332±1
16	40.5±0.1	141.9±0.05	16.8±0.1	18.46±0.04	341±1
17	41.5±0.1	142.9±0.05	17.8±0.1	18.33±0.04	336±1
18	42.5±0.1	141.9±0.05	18.8±0.1	18.46±0.04	341±1
19	43.5±0.1	142.8±0.05	19.8±0.1	18.35±0.04	337±1
20	44.5±0.1	144.3±0.05	20.8±0.1	18.16±0.04	330±1
21	45.5±0.1	142.2±0.05	21.8±0.1	18.42±0.04	339±1
22	46.5±0.1	139.8±0.05	22.8±0.1	18.74±0.04	351±1
23	47.5±0.1	136.7±0.05	23.8±0.1	19.17±0.04	368±1
24	48.5±0.1	133.0±0.05	24.8±0.1	19.70±0.04	388±1
25	49.5±0.1	129.5±0.05	25.8±0.1	20.23±0.04	409±1
26	50.5±0.1	125.7±0.05	26.8±0.1	20.84±0.04	434±1
27	51.5±0.1	124.3±0.05	27.8±0.1	21.08±0.04	444±1
28	52.5±0.1	123.4±0.05	28.8±0.1	21.23±0.04	451±1
29	53.5±0.1	120.9±0.05	29.8±0.1	21.67±0.04	470±1
30	54.5±0.1	117.5±0.05	30.8±0.1	22.30±0.04	497±1
31	55.5±0.1	114.0±0.05	31.8±0.1	22.98±0.04	528±1
32	56.5±0.1	111.2±0.05	32.8±0.1	23.56±0.05	555±2
33	57.5±0.1	110.5±0.05	33.8±0.1	23.71±0.05	562±2
34	58.5±0.1	108.1±0.05	34.8±0.1	24.24±0.05	588±2
35	59.5±0.1	107.1±0.05	35.8±0.1	24.46±0.05	598±2
36	60.5±0.1	104.6±0.05	36.8±0.1	25.05±0.05	628±2
37	61.5±0.1	102.1±0.05	37.8±0.1	25.66±0.05	658±2
38	62.5±0.1	100.1±0.05	38.8±0.1	26.17±0.05	685±2
39	63.5±0.1	99.6±0.05	39.8±0.1	26.31±0.05	692±2
40	64.5±0.1	97.3±0.05	40.8±0.1	26.93±0.05	725±2
41	65.5±0.1	95.8±0.05	41.8±0.1	27.35±0.05	748±2
42	66.5±0.1	94.7±0.05	42.8±0.1	27.67±0.05	766±2
43	67.5±0.1	94.0±0.05	43.8±0.1	27.87±0.06	777±2
44	68.5±0.1	92.9±0.05	44.8±0.1	28.20±0.06	795±2
45	69.5±0.1	91.1±0.05	45.8±0.1	28.76±0.06	827±2

where [a] h_1 is the reading of the top position of the weight before it starts to fall,

[b] h is the distance of fall of the weight which is obtained by $h = h_1 - h_2 + d/2$,

h_2 (= $(25\pm0.05)\times10^{-2}$ m) is the top position of the weight at the start of blocking of the photogate,

d (= $(2.62\pm0.005)\times10^{-2}$ m) is the length of the weight, and

[c] v is obtained from $v = d/\Delta t$.

3. Relate the coefficient C to the parameters of the MBB. (1.0 points)

$$h = C v^2, \text{ where } C = \{m_o + I/R^2 + m(l^2 + 2/5\, r^2)/R^2\}/2m_o g$$

(Explanation) The ball is at static equilibrium ($x = l$). When the speed of the weight is v, the increase in kinetic energy of the whole system is given by

$$\begin{aligned}\Delta K &= 1/2\, m_o v^2 + 1/2\, I\omega^2 + 1/2\, m(l^2 + 2/5\, r^2)\omega^2 \\ &= 1/2\, \{m_o + I/R^2 + m(l^2 + 2/5\, r^2)/R^2\}v^2,\end{aligned}$$

where ω (= v/R) is the angular velocity of the Mechanical "Black Box" and I is the *effective* moment of inertia of the whole system except the ball. Since the decrease in gravitational potential energy of the weight is

$$\Delta U = -\, m_o g h\ ,$$

the energy conservation ($\Delta K + \Delta U = 0$) gives

$$\begin{aligned}h &= 1/2\, \{m_o + I/R^2 + m(l^2 + 2/5\, r^2)/\mathrm{R}^2\}v^2/m_o g \\ &= C v^2, \quad \text{where } C = \{m_o + I/R^2 + m(l^2 + 2/5\, r^2)/\mathrm{R}^2\}/2m_o g\end{aligned}$$

4. Relate the coefficients A and B to the parameters of the MBB. (1.0 points)

$$\begin{aligned}h = A v^2 + B, \text{ where } A &= [m_o + I/R^2 + m\{(L/2 - \delta - r)^2 + 2/5\, r^2\}/R^2]/2m_o g \\ \text{and } B &= [\, -k_1(\, L/2 - l - \delta - r)^2 \\ &\quad + k_2\{(L - 2\delta - 2r)^2 - (L/2 + l - \delta - r)^2\}]\, /2m_o g\end{aligned}$$

(Explanation) The ball stays at the end cap of the tube ($x = L/2 - \delta - r$). When the speed of the weight is v, the increase in kinetic energy of the whole system is given by

$$K = 1/2\ [m_o + I/R^2 + m\{(L/2 - \delta - r)^2 + 2/5\ r^2\}/R^2]v^2.$$

Since the increase in elastic potential energy of the springs is

$$\Delta U_e = 1/2\ [- k_1(L/2 - l - \delta - r)^2 + k_2\{(L - 2\delta - 2r)^2 - (L/2 + l - \delta - r)^2\}]\ ,$$

the energy conservation ($K + \Delta U + \Delta U_e = 0$) gives

$$h = 1/2\ [m_o + I/R^2 + m\{(L/2 - \delta - r)^2 + 2/5\ r^2\}/R^2]v^2/m_o g + \Delta U_e/m_o g = A\,v^2 + B,$$

where

$$A = [m_o + I/R^2 + m\{(L/2 - \delta - r)^2 + 2/5\ r^2\}/R^2]/2m_o g$$

and

$$B = [- k_1(L/2 - l - \delta - r)^2 + k_2\{(L - 2\delta - 2r)^2 - (L/2 + l - \delta - r)^2\}]\ /2m_o g.$$

5. Determine the value of m from your measurements and the results obtained in ***PART-A***. (3.0 points)

$$m = 6.2\times10^{-2}\ \text{kg}$$

(Explanation) From the results obtained in ***PART-B*** 3 and 4 we get

$$A - C = \frac{m}{2gm_o R^2}\left\{(L/2 - \delta - r)^2 - l^2\right\}.$$

The measured values are $L = (40.0\pm0.05)\times10^{-2}$ m

$m_o = (100.4\pm0.05)\times10^{-3}$ kg

$2R = (3.91\pm0.005)\times10^{-2}$ m

Therefore,

$$(L/2 - \delta - r)^2 = \{(20.0\pm0.03) - 0.5 - 1.1\}^2 \times10^{-4}\ \text{m}^2 = (338.6\pm0.8)\times10^{-4}\ \text{m}^2$$

and

$$2gm_oR^2 = 2\times980\times(100.4\pm0.05)\times(1.955\pm0.003)^2 \times10^{-6}\text{kg·m}^3/\text{s}^2 = (752\pm2)\times10^{-6}\text{kg·m}^3/\text{s}^2.$$

The slopes of the two straight lines in the graph (Fig. B-1) of ***PART-B*** 1 are

A = 5.0±0.1s^2/m and C = 2.4±0.1s^2/m,

respectively, and

A - C = 2.6±0.1s^2/m.

Since we already obtained $m\times l = (M+m)\times l_{cm}$ = 2.96×10^{-3}kg·m from ***PART-A***, the equation

$$(338.6\pm0.8)m^2 - (752\pm2)\times10^3\times(0.026\pm0.001)m - (296\pm8)^2 = 0$$

or

$$(338.6\pm0.8)m^2 - (19600\pm800)m - (88000\pm3000) = 0$$

is resulted, where m is expressed in the unit of g.
The roots of this equation are

$$m = \frac{(9800\pm400)\pm\sqrt{(9800\pm400)^2+(338.6\pm0.8)\times(88000\pm3000)}}{(338.6\pm0.8)}.$$

The physically meaningful positive root is

$$m = \frac{(9800\pm400)+\sqrt{(126000000\pm6000000)}}{(338.6\pm0.8)} = (62\pm2)\,\text{g} = (6.2\pm0.2)\times10^{-2}\,\text{kg}.$$

PART-C The spring constants k_1 *and* k_2 (6.0 points)

1. Measure the periods T_1 and T_2 of small oscillation shown in Figs. 3 (1) and (2) and write down their values, respectively. (1.0 points)

T_1 = 1.1090s and T_2 = 1.0193s

(Explanation)

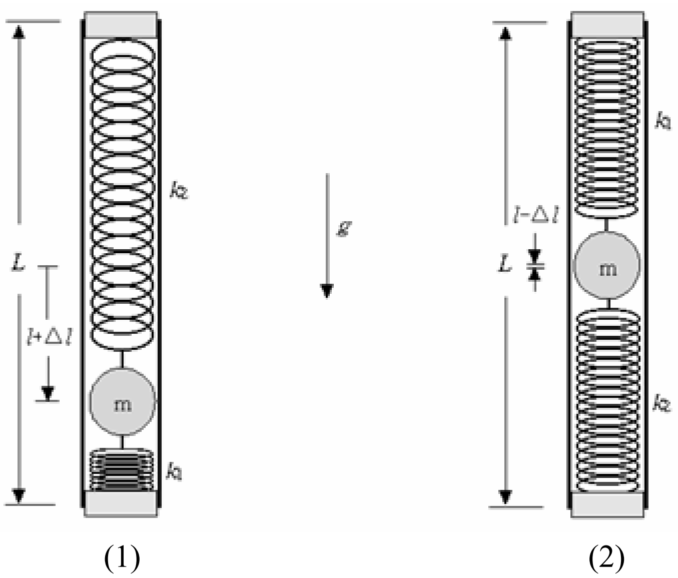

Fig. C-1 Small oscillation experimental set up

The measured periods are

	T_1 (s)		T_2 (s)
1	1.1085±0.00005	1	1.0194±0.00005
2	1.1092±0.00005	2	1.0194±0.00005
3	1.1089±0.00005	3	1.0193±0.00005
4	1.1085±0.00005	4	1.0191±0.00005
5	1.1094±0.00005	5	1.0192±0.00005
6	1.1090±0.00005	6	1.0194±0.00005
7	1.1088±0.00005	7	1.0194±0.00005
8	1.1090±0.00005	8	1.0191±0.00005
9	1.1092±0.00005	9	1.0192±0.00005
10	1.1094±0.00005	10	1.0193±0.00005

By averaging the10 measurements for each configuration, respectively, we get

$$T_1 = 1.1090 \pm 0.0003\text{s} \quad \text{and} \quad T_2 = 1.0193 \pm 0.0001\text{s}.$$

2. Explain, by using equations, why the angular frequencies ω_1 and ω_2 of small oscillation of the configurations are different. (1.0 points)

$$\omega_1 = \sqrt{\frac{Mg\,{}^{L}\!/_{2} + mg\left({}^{L}\!/_{2} + l + \Delta l\right)}{I_o + m\left\{\left({}^{L}\!/_{2} + l + \Delta l\right)^2 + \frac{2}{5}r^2\right\}}}$$

$$\omega_2 = \sqrt{\frac{Mg\,{}^{L}\!/_{2} + mg\left({}^{L}\!/_{2} - l + \Delta l\right)}{I_o + m\left\{\left({}^{L}\!/_{2} - l + \Delta l\right)^2 + \frac{2}{5}r^2\right\}}}$$

(Explanation) The moment of inertia of the Mechanical "Black Box" with respect to the pivot at the top of the tube is

$$I_1 = I_o + m\left\{\left({}^{L}\!/_{2} + l + \Delta l\right)^2 + \frac{2}{5}r^2\right\} \quad \text{or} \quad I_2 = I_o + m\left\{\left({}^{L}\!/_{2} - l + \Delta l\right)^2 + \frac{2}{5}r^2\right\}$$

depending on the orientation of the MBB as shown in Figs. C-1(1) and (2), respectively.

When the MBB is slightly tilted by an angle θ from vertical, the torque applied by the gravity is

$$\tau_1 = Mg\left({}^{L}\!/_{2}\right)\sin\theta + mg\left({}^{L}\!/_{2} + l + \Delta l\right)\sin\theta \approx \left\{Mg\left({}^{L}\!/_{2}\right) + mg\left({}^{L}\!/_{2} + l + \Delta l\right)\right\}\theta$$

or

$$\tau_2 = Mg\left({}^{L}\!/_{2}\right)\sin\theta + mg\left({}^{L}\!/_{2} - l + \Delta l\right)\sin\theta \approx \left\{Mg\left({}^{L}\!/_{2}\right) + mg\left({}^{L}\!/_{2} - l + \Delta l\right)\right\}\theta$$

depending on the orientation.

Therefore, the angular frequencies of oscillation become

$$\omega_1 = \sqrt{\frac{{}^{\tau_1}\!/_{\theta}}{I_1}} = \sqrt{\frac{Mg\,{}^{L}\!/_{2} + mg\left({}^{L}\!/_{2} + l + \Delta l\right)}{I_o + m\left\{\left({}^{L}\!/_{2} + l + \Delta l\right)^2 + \frac{2}{5}r^2\right\}}}$$

and

$$\omega_2 = \sqrt{\frac{{}^{\tau_2}\!/_{\theta}}{I_2}} = \sqrt{\frac{Mg\,{}^{L}\!/_{2} + mg\left({}^{L}\!/_{2} - l + \Delta l\right)}{I_o + m\left\{\left({}^{L}\!/_{2} - l + \Delta l\right)^2 + \frac{2}{5}r^2\right\}}}.$$

3. Evaluate Δl by eliminating I_o from the previous results. (1.0 points)

$$\Delta l = (7.2 \pm 0.9)\,\text{cm} = (7.2 \pm 0.9)\times 10^{-2}\,\text{m}$$

(Explanation) By rewriting the two expressions for the angular frequencies ω_1 and ω_2 as

$$Mg\, L/2 + mg\left(L/2 + l + \Delta l\right) = I_o\omega_1^2 + m\omega_1^2\left\{\left(L/2 + l + \Delta l\right)^2 + \frac{2}{5}r^2\right\}$$

and

$$Mg\, L/2 + mg\left(L/2 - l + \Delta l\right) = I_o\omega_2^2 + m\omega_2^2\left\{\left(L/2 - l + \Delta l\right)^2 + \frac{2}{5}r^2\right\}$$

one can eliminate the unknown moment of inertia I_o of the MBB without the ball.
By eliminating the I_o one gets the equation for Δl

$$\left(\omega_2^2 - \omega_1^2\right)\left\{\frac{(M+m)gL}{2} + mg\Delta l\right\} + \left(\omega_1^2 + \omega_2^2\right)mgl = \omega_1^2\omega_2^2 m(L + 2\Delta l)(2l).$$

From the measured or given values we get,

$$\left(\omega_2^2 - \omega_1^2\right) = \left\{\left(\frac{2\pi}{T_2}\right)^2 - \left(\frac{2\pi}{T_1}\right)^2\right\} = \left(\frac{6.2832}{1.0193 \pm 0.0001}\right)^2 - \left(\frac{6.2832}{1.1090 \pm 0.0003}\right)^2$$

$$= 5.90 \pm 0.01\,\text{s}^{-2}$$

$$\frac{(M+m)gL}{2} = \frac{(141.1 \pm 0.05)\times 980 \times (40.0 \pm 0.05)}{2} = (27.66 \pm 0.04)\times 10^{-2}\,\text{kg·m}^2/\text{s}^2$$

$$\left(\omega_1^2 + \omega_2^2\right)mgl = \left\{\left(\frac{2\pi}{T_1}\right)^2 + \left(\frac{2\pi}{T_2}\right)^2\right\}(M+m)l_{cm}g$$

$$= \left\{\left(\frac{6.2832}{1.1090 \pm 0.0003}\right)^2 + \left(\frac{6.2832}{1.0193 \pm 0.0001}\right)^2\right\}\times (296 \pm 8)\times 980$$

$$= (203 \pm 5) \times 10^{-2} \text{ kg·m}^2/\text{s}^4$$

$$\omega_1^2 \omega_2^2 ml = \left(\frac{2\pi}{T_1}\right)^2 \left(\frac{2\pi}{T_2}\right)^2 (M+m) l_{cm}$$

$$= \left(\frac{6.2832}{1.1090 \pm 0.0003}\right)^2 \left(\frac{6.2832}{1.0193 \pm 0.0001}\right)^2 \times (296 \pm 8)$$

$$= (3.6 \pm 0.1) \text{ kg·m/s}^4.$$

Therefore, the equation we obtained in ***PART-C*** 3 becomes

$$(5.90 \pm 0.01)\{(27.66 \pm 0.04) \times 10^5 + (62 \pm 2) \times 980 \times \Delta l\} + (203 \pm 5) \times 10^5$$

$$= (7.2 \pm 0.2) \times 10^5 \times \{(40.0 \pm 0.05) + 2\Delta l\},$$

where Δl is expressed in the unit of cm. By solving the equation we get

$$\Delta l = (7.2 \pm 0.9) \text{ cm} = (7.2 \pm 0.9) \times 10^{-2} \text{ m}$$

4. Write down the value of the effective total spring constant k of the two-spring system. (2.0 points)

$$k = 9 \text{ N/m}$$

(Explanation) The effective total spring constant is

$$k \equiv \frac{mg}{\Delta l} = \frac{(62 \pm 2) \times 980}{7.2 \pm 0.9} = 9000 \pm 1000 \text{ dyne/cm} \quad \text{or} \quad 9 \pm 1 \text{N/m}.$$

5. Obtain the respective values of k_1 and k_2. Write down their values. (1.0 point)

$k_1 = 5.7$ N/m

$k_2 = 3$ N/m

(Explanation) When the MBB is in equilibrium on a horizontal plane the force balance condition for the ball is that

$$\frac{L/2 - l - \delta - r}{L/2 + l - \delta - r} = \frac{N_1}{N_2} = \frac{k_2}{k_1}\,.$$

Since $k = k_1 + k_2$, we get

$$k_1 = \frac{k}{\dfrac{L/2 - l - \delta - r}{L/2 + l - \delta - r} + 1} = \frac{L/2 + l - \delta - r}{L - 2\delta - 2r}k$$

and

$$k_2 = k - k_1 = \frac{L/2 - l - \delta - r}{L - 2\delta - 2r}k\,.$$

From the measured or given values

$$\frac{L/2 + l - \delta - r}{L - 2\delta - 2r} = \frac{(20.0 \pm 0.03) + \left(\dfrac{296 \pm 8}{62 \pm 2}\right) - 0.5 - 1.1}{(40.0 \pm 0.05) - 1.0 - 2.2} = 0.63 \pm 0.005\,.$$

Therefore,

$k_1 = (0.63 \pm 0.005) \times (9000 \pm 1000) = 5700 \pm 600$ dyne/cm or 5.7±0.6N/m,

and

$k_2 = (9000 \pm 1000) - (5700 \pm 600) = 3000 \pm 1000$ dyne/cm or 3±1N/m.