SOLUTION T3 : A Heavy Vehicle Moving on An Inclined Road

To simplify the model we use the above figure with $h_1 = h + 0.5 \ t$

$R_0 = R$

1. Calculation of the moment inertia of the cylinder

$R_i = 0.8 \ R_o$

Mass of cylinder part : $m_{cylinder} = 0.8 \ M$

Mass of each rod : $m_{rod} = 0.025 \ M$
2. Force diagram and balance equations:

To simplify the analysis we divide the system into three parts: frame (part 1) which mainly can be treated as flat homogeneous plate, rear cylinders (two cylinders are treated collectively as part 2 of the system), and front cylinders (two front cylinders are treated collectively as part 3 of the system).

Part 1: Frame

The balance equation related to the forces work to this parts are:

\[
I = \int r^2 dm = \int_{\text{cyl.shell}} r^2 dm + \int_{\text{rod1}} r^2 dm + \ldots + \int_{\text{rod n}} r^2 dm
\]

\[
\int_{\text{cyl.shell}} r^2 dm = 2\pi \int_{R_i}^{R_o} r^3 dr = 0.5\pi \pi (R_o^4 - R_i^4) = 0.5m_{\text{cylinder}} (R_o^2 + R_i^2)
\]

\[
= 0.5(0.8M)R^2 (1 + 0.64) = 0.656MR^2
\]

\[
\int_{\text{rod}} r^2 dm = \lambda \int_0^{R_{in}} r^2 dr = \frac{1}{3} \lambda R_{in}^3 = \frac{1}{3} m_{\text{rod}} R_{in}^2 = \frac{1}{3} 0.025M (0.64R^2) = 0.00533MR^2
\]

The moment inertia of each wheel becomes

\[
I = 0.656MR^2 + 8 \times 0.00533MR^2 = 0.7MR^2
\]

0.4 pts
Required conditions:

Balance of force in the horizontal axis

\[ m_1 g \sin \Theta - f_{12h} - f_{13h} = m_1 a \]  (1) 0.2 pts

Balance of force in the vertical axis

\[ m_1 g \cos \Theta = N_{12} + N_{13} \]  (2) 0.2 pts

Then torsion against O is zero, so that

\[ N_{12}l - N_{13}l + f_{12h}h_1 + f_{13h}h_1 = 0 \]  (3) 0.2 pts

Part two: Rear cylinder

\[ f_2 \]

\[ N_2 \]

\[ f_{21h} \]

\[ N_{21} \]

\[ Mg \]

From balance condition in rear wheel:

\[ f_{21h} - f_2 + Mg \sin \Theta = Ma \]  (4) 0.15 pts

\[ N_2 - N_{21} - Mg \cos \Theta = 0 \]  (5) 0.15 pts

For pure rolling:

\[ f_2 R = I\alpha_2 = I \frac{a_2}{R} \]

or \[ f_2 = \frac{I}{R^2} a \]  (6)

For rolling with sliding:

\[ F_2 = u_k N_2 \]  (7) 0.2 pts

Part Three: Front Cylinder:
From balance condition in the front wheel:

\[ f_{31h} - f_3 + Mg \sin \theta = Ma \quad (8) \quad 0.15 \text{ pts} \]

\[ N_3 - N_{31} - Mg \cos \theta = 0 \quad (9) \quad 0.15 \text{ pts} \]

For pure rolling:

\[ f_3 R = I \alpha_3 = I \frac{a_3}{R} \]

or \[ f_3 = \frac{I}{R^2} a \quad (10) \]

For rolling with sliding:

\[ F_3 = u_k N_3 \quad (11) \]

3. From equation (2), (5) and (9) we get

\[ m_1 g \cos \theta = N_2 - m_2 g \cos \theta + N_3 - m_3 g \cos \theta \]

\[ N_2 + N_3 = (m_1 + m_2 + m_3) g \cos \theta = 7Mg \cos \theta \quad (12) \]

And from equation (3), (5) and (8) we get

\[ (N_3 - Mg \cos \theta) l - (N_2 - Mg \cos \theta) l = h_1 (f_2 + Ma - Mg \sin \theta + f_3 + Ma - Mg \sin \theta) \]

\[ (N_3 - N_2) = h_1 (f_2 + 2Ma - 2Mg \sin \theta + f_3) l \]

Equations 12 and 13 are given \textbf{0.25 pts}

\textbf{CASE ALL CYLINDER IN PURE ROLLING}

From equation (4) and (6) we get
\[ f_{21h} = \left( I/R^2 \right) a + Ma - Mg \sin \theta \]  
(14) 0.2 pts

From equation (8) and (10) we get
\[ f_{31h} = \left( I/R^2 \right) a + Ma - Mg \sin \theta \]  
(15) 0.2 pts

Then from eq. (1), (14) and (15) we get
\[ 5Mg \sin \theta - \left\{ \left( I/R^2 \right) a + Ma - Mg \sin \theta \right\} - \left\{ \left( I/R^2 \right) a + Ma - Mg \sin \theta \right\} = m_1 a \]

\[ 7 \text{Mg} \sin \theta = 2 \left( I/R^2 + 7M \right) a \]
\[ a = \frac{7 \text{Mg} \sin \Theta}{7M + 2 \frac{I}{R^2}} = \frac{7 \text{Mg} \sin \Theta}{7M + 2 \frac{0.7MR^2}{R^2}} = 0.833 \text{g} \sin \Theta \]  
(16) 0.35 pts

\[ N_3 = \frac{7M}{2} g \cos \Theta + \frac{h_1}{l} \left[ \left( M + \frac{I}{R^2} \right) \times 0.833 \text{g} \sin \Theta - Mg \sin \Theta \right] \]
\[ = 3.5 \text{Mg} \cos \Theta + \frac{h_1}{l} \left[ \left( M + 0.7M \right) \times 0.833 \text{g} \sin \Theta - Mg \sin \Theta \right] \]
\[ = 3.5 \text{ Mg} \cos \Theta + 0.41 \frac{h_1}{l} Mg \sin \Theta \]

\[ N_2 = \frac{7M}{2} g \cos \Theta - \frac{h_1}{l} \left[ \left( \frac{I}{R^2} + M \right) \times 0.833 \text{g} \sin \Theta - Mg \sin \Theta \right] \]
\[ = 3.5g \cos \Theta - \frac{h_1}{l} \left[ (0.7M + M) \frac{7 \text{Mg} \sin \Theta}{0.7M + 7M} - 2Mg \sin \Theta \right] \]
\[ = 3.5g \cos \Theta - 0.41 \frac{h_1}{l} Mg \sin \Theta \]

The Conditions for pure rolling:
\[ f_x \leq \mu_s N_2 \quad \text{and} \quad f_y \leq \mu_s N_3 \]
\[ \frac{I_2}{R_2^2} a \leq \mu_s N_2 \quad \text{and} \quad \frac{I_3}{R_3^2} a \leq \mu_s N_3 \]  
0.2 pts

The left equation becomes
\[ 0.7M \times 0.833g \sin \theta \leq \mu_s (3.5 \text{Mg} \cos \Theta - 0.41 \frac{h_1}{l} Mg \sin \theta) \]
\[ \tan \theta \leq \frac{3.5 \mu_s}{0.5831 + 0.41 \mu_s \frac{h_1}{l}} \]
While the right equation becomes

\[ 0.7m \times 0.833g \sin \theta \leq \mu_j (3.5mg \cos \theta + 0.41 \frac{h_i}{l} mg \sin \theta) \]

\[ \tan \theta \leq \frac{3.5\mu_j}{0.5831 - 0.41\mu_j \frac{h_i}{l}} \]  

(17)  

0.1 pts

**CASE ALL CYLINDER SLIDING**

From eq. (4)  \( f_{21h} = Ma + u_k N_2 - Mg \sin \theta \)  
(18)  

0.15 pts

From eq. (8)  \( f_{31h} = Ma + u_k N_3 - Mg \sin \theta \)  
(19)  

0.15 pts

From eq. (18) and (19):

\[ 5Mg \sin \theta - (Ma + u_k N_2 - Mg \sin \theta) - (Ma + u_k N_3 - Mg \sin \theta) = ma \]

\[ a = \frac{7Mg \sin \theta - \mu_k N_2 - \mu_k N_3}{7M} = g \sin \theta - \frac{\mu_k (N_2 + N_3)}{7M} \]  

(20)  

0.2 pts

\[ N_3 + N_2 = 7Mg \cos \theta \]

From the above two equations we get:

\[ a = g \sin \theta - \mu_k g \cos \theta \]  

0.25 pts

The Conditions for complete sliding: are the opposite of that of pure rolling

\[ f_{2} \mu_j N'_2 \]  and  \[ f_{3} \mu_j N'_3 \]

\[ \frac{1}{R_2^2} a \mu_j N'_2 \]  and  \[ \frac{1}{R_3^2} a \mu_j N'_3 \]  

(21)  

0.2 pts

Where \( N_2' \) and \( N_3' \) is calculated in case all cylinder in pure rolling.  

0.1 pts

Finally we get

\[ \tan \theta \leq \frac{3.5\mu_j}{0.5831 + 0.41\mu_j \frac{h_i}{l}} \]  

and  \[ \tan \theta \leq \frac{3.5\mu_j}{0.5831 - 0.41\mu_j \frac{h_i}{l}} \]  

0.2 pts

The left inequality finally become decisive.

**CASE ONE CYLINDER IN PURE ROLLING AND ANOTHER IN SLIDING CONDITION**

{ For example \( R_3 \) (front cylinders) pure rolling while \( R_2 \) (Rear cylinders) sliding}
From equation (4) we get

\[ F_{21h} = m_2 a + u_k N_2 - m_2 g \sin \theta \]  

(22)  

0.15 pts

From equation (5) we get

\[ f_{31h} = m_3 a + (I/R^2) a - m_3 g \sin \theta \]  

(23)  

0.15 pts

Then from eq. (1), (22) and (23) we get

\[ m_1 g \sin \theta - \{ m_2 a + u_k N_2 - m_2 g \sin \theta \} - \{ m_3 a + (I/R^2) a - m_3 g \sin \theta \} = m_1 a \]

\[ m_1 g \sin \theta + m_2 g \sin \theta + m_3 \sin \theta - u_k N_2 = (I/R^2 + m_3) a + m_2 a + m_1 a \]

\[ 5Mg \sin \theta + Mg \sin \theta + Mg \sin \theta - u_k N_2 = (0.7M + M) a + Ma + 5Ma \]

\[ a = \frac{7Mg \sin \theta - \mu_k N_2}{7.7M} = 0.9091 g \sin \theta - \frac{\mu_k N_2}{7.7M} \]  

(24)  

0.2 pts

\[ N_3 - N_2 = \frac{h_i}{l} (\mu_k N_2 + \frac{I}{R^2} a + 2Ma - 2Mg \sin \Theta) \]

\[ N_3 - N_2 = \frac{h_i}{l} (\mu_k N_2 + 2.7M \times 0.9091 g \sin \Theta - 2.7\mu_k N_2 / 7.7 - 2Mg \sin \Theta) \]

\[ N_3 - N_2 (1 + 0.65\mu_k \frac{h_i}{l}) = 0.4546Mg \sin \Theta \]

\[ N_3 + N_2 = 7Mg \cos \theta \]

Therefore we get

\[ N_2 = \frac{7Mg \cos \Theta - 0.4546Mg \sin \Theta}{2 + 0.65\mu_k \frac{h_i}{l}} \]  

\[ N_3 = 7Mg \cos \Theta - \frac{7Mg \cos \Theta - 0.4546Mg \sin \Theta}{2 + 0.65\mu_k \frac{h_i}{l}} \]  

(25)  

0.3 pts

Then we can substitute the results above into equation (16) to get the following result

\[ a = 0.9091 g \sin \theta - \frac{\mu_k N_2}{7.7M} = 0.9091 g \sin \theta - \frac{\mu_k 7g \cos \theta - 0.4546g \sin \Theta}{2 + 0.65\mu_k \frac{h_i}{l}} \]  

(26)  

0.2 pts
The Conditions for this partial sliding is:

\[ f_2 \leq \mu_i N'_2 \quad \text{and} \quad f_3 \leq \mu_i N'_3 \]

\[ \frac{1}{R^2} a \leq \mu_i N'_2 \quad \text{and} \quad \frac{1}{R^2} a \leq \mu_i N'_3 \]  \hspace{1cm} (27) \hspace{1cm} 0.25 \text{ pts}

where \( N'_2 \) and \( N'_3 \) are normal forces for pure rolling condition

4. Assumed that after rolling \( d \) meter all cylinder start to sliding until reaching the end of incline road (total distant is \( s \) meter). Assumed that \( t_1 \) meter is reached in \( t_1 \) second.

\[ v_{n1} = v_o + at_1 = 0 + a_1 t_1 \]

\[ d = v_o t_1 + \frac{1}{2} a_1 t_1^2 = \frac{1}{2} a_1 t_1^2 \]

\[ t_1 = \sqrt{\frac{2d}{a_1}} \] \hspace{1cm} 0.5 \text{ pts}

\[ v_{n1} = a_1 \sqrt{\frac{2d}{a_1}} = \sqrt{2da_1} = \sqrt{2d0.833g \sin \Theta} = \sqrt{1.666dg \sin \Theta} \]  \hspace{1cm} (28)

The angular velocity after rolling \( d \) meters is same for front and rear cylinders:

\[ \frac{\omega_1}{R} = \frac{v_{n1}}{R} = \frac{1}{R} \sqrt{1.666dg \sin \Theta} \]  \hspace{1cm} (29) \hspace{1cm} 0.5 \text{ pts}

Then the vehicle sliding until the end of declining road. Assumed that the time needed by vehicle to move from \( d \) position to the end of the declining road is \( t_2 \) second.

\[ v_{t2} = v_{n1} + a_2 t_2 = \sqrt{1.666dg \sin \Theta} + a_2 t_2 \]

\[ s - d = v_{n1} t_2 + \frac{1}{2} a_2 t_2^2 \]

\[ t_2 = \frac{v_{n1} + \sqrt{v_{n1}^2 + 2a_2(s - d)}}{a_2} \]  \hspace{1cm} (30) \hspace{1cm} 0.4 \text{ pts}

\[ v_{t2} = \sqrt{1.666dg \sin \Theta} - v_{n1} + \sqrt{v_{n1}^2 + 2a_2(s - d)} \]

Inserting \( v_{n1} \) and \( a_2 \) from the previous results we get the final results.

For the angular velocity, while sliding they receive torsion:
\[ \tau = \mu_c NR \]
\[ \alpha = \frac{\tau}{I} = \frac{\mu_c NR}{I} \]

\[ \omega_{r2} = \omega_{i2} + \alpha t_2 = \frac{1}{R} \sqrt{1.666 \, d \, g \, \sin \theta} + \frac{\mu_c NR - v_{i1} + \sqrt{v_{i1}^2 + 2a_2 (s - d)}}{a_2} \]

(31)

0.6 pts